In the diagram above, angle $\alpha$ is 26.5° and angle $\beta$ is 63.5°. (Measure them to make sure). The length of side $b$ is 5.1 cm and the length of side $a$ is 2.54 cm. The angle with the square in it is 90°, making this a right triangle (any triangle having a 90° angle is a right triangle). Notice that the sum of all the angles in the triangle is 180°. Also, in a right triangle, the length of the longest side (called the hypotenuse and labeled here with a “c”) can be found in a simple way by looking at the length of the other two sides. Using the Pythagorean theorem, we find that the lengths are related by

$$a^2 + b^2 = c^2$$

There are other relationships between the sides and angles of a right triangle which we’ll need to know. These were discovered thousands of years ago – once we look at them, we’ll examine why we need to know about them. The accepted way to refer to the sides of a right triangle (relative to one of the angles) is to call them the opposite side (in the picture above, side $a$ is opposite from angle $\alpha$, and side $b$ is opposite from angle $\beta$), the adjacent side ($b$ is adjacent to angle $\alpha$ and $a$ is adjacent to angle $\beta$), and the hypoteneuse (side $c$, no matter what angle you’re talking about).

The Sine of an angle is equal to the length of the side opposite it divided by the hypoteneuse. In other words, we write

$$\sin \alpha = \frac{a}{c} \ OR \ \sin \beta = \frac{b}{c}$$

and you’ll hear people say “Sine is opposite over hypoteneuse”. The Cosine of an angle is similarly related, but it’s the length of the side adjacent to the angle divided by the length of the hypoteneuse:

$$\cos \alpha = \frac{b}{c} \ OR \ \cos \beta = \frac{a}{c}$$
If you look, you’ll notice that this can only be right if the Sine of $\beta$ is equal to the Cosine of $\alpha$ and vice versa. In fact, this is true. Since you also know that, for a right triangle, $\alpha+\beta = 90^\circ$, this means that the Cosine of any angle is equal to the Sine of $(90^\circ - \text{that angle})$. Finally, the relation between the sides opposite to and adjacent to an angle is called the Tangent:

$$\tan \alpha = \frac{a}{b} \quad \text{OR} \quad \tan \beta = \frac{b}{a}$$

There are various ways to remember these relationships – one is that Sine, Cosine, and Tangent are found by “Oscar Had A Heap Of Apples” → Sine = O/H, Cos = A/H, Tan = O/A where O, A, and H = Opposite, Adjacent, and Hypoteneuse.

Having gotten all this background out of the way, why do we care? It’s because we can frequently make things easier on ourselves if we break a two- (or three-) dimensional thing into two (or three) one-dimensional things. This is called finding the **components** of the two (or three) dimensional object. For example, look at the map below and assume you’re starting at the corner of maple & 7th streets and trying to get to the corner of elm & 4th streets.

If you’re in a car, you’ll probably drive West down Maple 3 blocks until you get to 4th, and then drive North for 2 blocks. You could also zigzag through the streets, or go North first, but your final path can be described as saying that you moved 2 blocks North and 3 blocks West.

If you were in a helicopter, would you take the same path? You’d probably move in a straight line from start to finish, just because you can. Your endpoint would be the same (as would your **displacement**), but your path might be different. How could you explain where you wanted to go to the helicopter pilot? You would give the pilot an angle (probably measured from the North, but not necessarily) and a distance to travel. The distance would be given by the Pythagorean theorem as
\[ \text{dist}^2 = (3 \text{ blocks})^2 + (2 \text{ blocks})^2 \]

or, more simply,

\[ \text{dist} = \sqrt{(3 \text{ blocks})^2 + (2 \text{ blocks})^2} = \sqrt{9 \text{ blocks}^2 + 4 \text{ blocks}^2} = \sqrt{13 \text{ blocks}^2} = 3.61 \text{ blocks} \]

Notice that the distance unit (blocks, in this case) remained squared even after the number of blocks was squared. We have to have blocks² inside the square root sign so that we’ll get just plain blocks out after taking the square root. (Really, blocks are a measure of area and not length, but let’s ignore that right now).

We know how far the helicopter has to travel, and we know it’s Northwest (just by looking at the map), but we have to find the precise angle. To do this, we have to decide which direction we’ll measure the angle from. We could pick any direction we want as our zero point, so let’s use West.

Notice that if we connected the West & North path the car took with the straight line path the helicopter should take, we’d get something like the first triangle we drew (Maple street is side \( b \) and 4th street is side \( a \)). The angle we’re looking for is just \( \alpha \). (Keep in mind that the street distances are not the same as the lengths of sides \( a \) and \( b \), so the angle will be different, but the procedure is the same).

We can use

\[ \tan \alpha = \frac{a}{b} = \frac{2 \text{ blocks}}{3 \text{ blocks}} = 0.67 \]

This means \( \alpha \) is 33.8°. Years ago, you would have had to look this value up in tables, but your calculator can do it for you now. What you need to do is find something that reverses (more accurately, inverts) the tangent operation. This is usually shown on your calculator as \( \tan^{-1} \) where the superscript (-1) means inverse. If you get 0.59 as an answer, your calculator is working in radians instead of degrees. If you get 37.6, you’re working in grads. There’s nothing wrong with these units, but if you ask your calculator for radians and put them down with a degree sign next to them, your answer will be wrong.

Since we found the angle labeled as \( \alpha \), we’re measuring the number of degrees North of West we would have to tell the pilot to aim. We could find the angle \( \beta \) just as easily, although once we know \( \alpha \), all we really have to do is take 90° - \( \alpha \) which would tell us that \( \beta \) is 56.2°. In other words, we could also tell the pilot to aim 56.2° West of North, which is the same direction as 33.8° North of West.
What we just saw was the way to combine two one-dimensional objects (the distances down the streets) into a two-dimensional object (distance + direction together, neither of which would mean much alone).

It’s just as likely that we would need to break a two (or three) dimensional vector (quantity with both magnitude and direction – our helicopter path has both) into its components along perpendicular directions. We would see this if the helicopter had moved from start to finish first and wanted to tell someone in a car how to meet them.

This vector will be given to us as a magnitude and a direction. The magnitude is always the hypoteneuse of a triangle. It has to be – the hypoteneuse is always the longest side of a triangle, and the vector is always at least as long as each of its components (a small line can’t be made up of two perpendicular larger lines!).

The components are the two sides which meet in a right angle (components are always perpendicular – going East or West does not change your position North or South at all). We customarily call the two perpendicular directions x and y (for no particular reason, and there’s absolutely no physical significance in our choice). Usually, the x-axis tends to be drawn from left to right, and the y-axis goes up and down (again, just a habit). Look at the vector below (labeled with magnitude and direction)

We want to break it into left-right (x) and up-down (y) components. We draw the only triangle that has all of these characteristics (left-right side & up-down side forming a right angle, with their other ends connected by a 4 cm line at a 50° angle):
How do we find the lengths of sides x & y? We use the same trig formulas we’ve been using. We know that the **hypoteneuse** is 4 cm. If we decide to find side y first, it’s the side **opposite** a 50° angle. The function that combines the opposite side & hypoteneuse is Sine. This means

\[
\text{Sin } 50° = \frac{\text{opposite}}{\text{hypoteneuse}} = \frac{y}{4 \text{ cm}}
\]

We can quickly solve this for y and find that the length of y must be 4 cm * Sin 50°, which is 3.06 cm. We can use the same trick to find the x component. If we want to use Sin, we need to know the third angle (easy enough – it’s just 90° - 50° = 40°). It’s just as easy to use Cosine and our original angle and get

\[
\text{Cos } 50° = \frac{\text{adjacent}}{\text{hypoteneuse}} = \frac{x}{4 \text{ cm}}
\]

The x component is therefore 4 cm * Cos 50° = 2.57 cm. We can square the sides, add them, and get the square of the length of the hypoteneuse, if we want to check this. Except for small rounding errors, it works.

One thing you should **not** do is just memorize “The x-component is always = Vector length * Cos of angle”. You can arrange things so that this is true, but if someone else (me) draws & labels the triangle, it may be that Sine is the trig function you need to find the x component. If you understand the triangle you’re making, and you remember Sine = O/H, Cosine = A/H, and Tangent = O/A, you’ll never go wrong.

Why is breaking vectors into components so important? Because adding vectors becomes trivial if you can break them into parts first. All you have to do is add the x parts together, and (separately) add the y parts together. Take your new total x and new total y and use the methods above to reconstruct a new final vector. Also, we’ll see that some things, like projectile motion, are made **vastly** simpler by breaking motion into components along the ground and perpendicular to it. How can this be if I said earlier that there is no physical significance to your choice of starting direction? The physics will work no matter what – there really is no physics in your decision to make x go left-right rather than up-down (or lower left – upper right). However, you will see as we go on that you can make your work much harder or much easier by a careful choice of coordinate directions. In the same way, you could decide to do your taxes by converting dollars to francs, francs to pounds, and pounds to rubles, then converting everything back at the end. You just wouldn’t want to do it!