Atomic Physics & the Electromagnetic Spectrum

The earliest X-ray of the human body was taken by Wilhelm Röntgen in 1895 when he noticed that the bones in his hand became visible after passing in front of a cathode ray tube. Although the understanding of the nature and source of X-rays was still primitive, it was clear that the ability to see inside the body without surgery was a major medical breakthrough.

To understand more about the physics involved, we need to know about the electromagnetic spectrum and waves. The different colors of visible light that we can see represent different wavelengths (and frequencies) of the changes in the magnetic and electric fields that make up light. The wavelength is the separation in space between two peaks (or valleys) of the wave, while the period is the separation in time between them. The frequency is just 1/period, and is measured in Hertz (Hz) if the period is in seconds. The formula connecting wavelength, frequency, and wave speed is just

\[ f \lambda = c \]

where \( f \) is the frequency of the light, \( \lambda \) is its wavelength, and \( c \) represents the wave's speed. For visible light, the wavelengths range from about 380 nm (billionths of a meter) for violet to about 750 nm (red). That means the light’s frequency ranges from \( 4 \times 10^{14} \) Hz (red) to about \( 7.9 \times 10^{14} \) Hz (violet).

The visible part of the spectrum is only a tiny slice of what’s out there. Moving from visible light to lower frequency electromagnetic radiation, we get to infrared radiation first. This is sensed (if it’s intense enough) as heat, and it’s what warms your hand if you put it near a light bulb. It’s also what allows police departments & soldiers to see people in total darkness (night vision goggles), and it’s what your remote control puts out. How far does this part of the spectrum reach? There are no hard physical boundaries between any of the regions of the spectrum – some are more definite than others because of biology (we can see red, we can’t see infrared) and some by regulations (the FCC or some international standards body determines that FM radio runs from 88 MHz to 108 MHz). As far as the physics is concerned, it’s a smooth transition from low frequency to high frequency.

As we go to frequencies in the low end of the infrared, we eventually get to the microwave region (around \( 10^9 – 10^{11} \) Hz or so). The microwaves that cook your food are in this range, as are most cordless phones, cell phones, and wireless computer networking equipment. Microwaves are sometimes lumped into what’s called the “radio region” of the spectrum. Also part of that is the FM range mentioned above. Broadcast TV (which covers a much larger portion of the spectrum than the 20 MHz or so allotted to FM radio) starts in the FM region and reaches up into the microwave region (you can see this for yourself if you’re ever driving through an area that has a local TV station broadcasting as channel 6 (not necessarily cable channel 6) – you’ll be able to pick up the audio on the low end of your FM radio).
Even lower in frequency, we get to AM radio, which runs from about 550 kHz to about 1650 KHz or so. Below this, there are specialized transmitters which can penetrate deeper into the ocean to communicate with submarines without requiring them to surface. Also at the low end of the spectrum is the 60 Hz radiation produced by power lines (and just about everything plugged into them).

What's above the violet end of the visible spectrum? Immediately above it is the ultraviolet range. This is responsible for giving people tans (and skin cancer), and reaches up to about $10^{17}$ Hz. Above that, up to around $10^{20}$ Hz or so, we find X-rays. Beyond that, the rest of the way up, we just talk about gamma rays. We'll see more about them later.

Not too many years after the discovery of X-rays came the idea that light might have a smallest size associated with it. People had known for quite some time that matter has a basic unit known as the atom. There are different kinds of atom, each with a different mass. We don’t generally notice this since the atom has such a ridiculously small mass, but there is a difference in the smallest mass of iron you can have compared to the smallest mass of oxygen you can have. For the most common variety of oxygen, that smallest mass (the mass of the atom) is about $2.7 \times 10^{-27}$ kg. For iron, it’s about $9.4 \times 10^{-27}$ kg. There is simply no way to get $2.7 \times 10^{-27}$ kg of iron.

The similarity between light and matter is that the smallest piece of light (known as a photon) has an energy which depends on its frequency. The most convenient unit of energy for this region of physics is known as the electron Volt, or eV. If an electron goes from one end of an AA battery (1.5 Volts when new) to the other, it will gain 1.5 eV of energy. The relationship between energy and frequency for a photon can be written simply as

$$E = h f$$

where $E$ is the photon’s energy in eV, $f$ is the frequency of the light in Hz, and $h$ is a constant (known as Planck’s constant) which has a value of $4.14 \times 10^{-15}$ eV sec.

This tells us that the smallest amount of red light we can have will have an energy of about 1.7 eV, while the smallest amount of violet light would have an energy of about 3.5 eV. Just as we can’t have $3 \times 10^{-27}$ kg of iron, we can’t have 2 eV of violet light.

X-rays have much smaller wavelengths (and therefore much higher frequencies = much higher energies) than visible light. Medical X-rays will typically be in the range of 20,000 eV (usually written as 20 keV) to about 100 keV. These high-energy photons can be dangerous (a fact that wasn’t appreciated when shoe stores used to use powerful X-ray machines to look inside children’s shoes to check the fit!) because of their effect on matter.

To understand the potential dangers of X-rays (grouped with ultraviolet radiation and gamma rays as ionizing radiation), we need to look at atoms in detail. The periodic table of elements lists over 100 atoms discovered so far. In an atom, more than 99.9%
of the mass is concentrated at the center in the nucleus. The nucleus is composed of positively charged protons and uncharged neutrons. In terms of physical size, the nucleus has a diameter only about 1/100,000th that of the whole atom, for a volume of about $10^{-15}$ the atom’s volume. If the atom were an Olympic-size swimming pool, the nucleus would be smaller than a sesame seed (but would still contain about 99.98% of the mass).

The number of protons in the nucleus gives the element’s identity – hydrogen has one, helium has two, etc. The number of neutrons is a little more variable, for reasons we’ll see later. A hydrogen atom can have zero, one, or two neutrons in its nucleus. We would call each of these a different isotope of hydrogen. In each case, the nucleus would have the same charge (which would just be the charge on a single proton) but its mass would increase, since a neutron has almost the same mass as a proton.

In general, there are as many electrons around the nucleus as there are protons inside it. This means that the atom as a whole is electrically neutral. The negatively charged electrons stay near the nucleus because of the mutual attraction between them and the positively charged protons. Just as you have to expend energy to separate two magnets which are stuck together, it takes energy to remove one or more of the electrons from the atom. This process is called ionization because the neutral atom is replaced by a free electron and an ion.

Photons of visible light frequencies or less are generally unable to cause this electron removal. The reason ultraviolet radiation is harmful is because it crosses this threshold and can create ions. While some ions are naturally present and necessary in the body, large numbers of them in random locations can cause problems such as cancer.

Radiation from the near UV (ultraviolet, but not very far from the visible region) may only create one ion when it hits the body because its energy is just over the minimum needed for ionization. X-rays and gamma rays are so much more energetic that each photon may create many ions before finally being absorbed.

**X-ray Origins**

Before we can design a machine to produce X-rays reliably, we need to know what’s happening on the microscopic level. The electrons in an atom are uniquely identified by their quantum numbers. These numbers describe the energy and general range of motion of the electrons. The different energies of the electrons tell us how much energy it would take to ionize the atom by removing that electron. Other quantum numbers describe things like the way an applied magnetic field affects where an electron spends most of its time. Finally, we can consider the electron to be like a tiny magnet which can be oriented with the north pole up or down (called spin-up or spin-down states).

The Pauli exclusion principle tells us that no two electrons in the same atom can have the same quantum numbers (including spin). This is what makes matter stable – if not for this fact, all of the electrons in any atom would immediately move down to the ground state. As it is, electrons can only move into unoccupied states. The attraction
between the nucleus and the electron means that an electron bound in an atom has **negative** energy. We say the energy is negative because it would take the input of energy to free the electron and move it out to infinity, where the energy of attraction between it and the nucleus would be zero.

If an electron wants to move from a lower state (one in which the electron is, on average, closer to the nucleus) to a higher state (more distant on average), it can only do that if energy is sent in from outside. This small amount of energy is just a photon. Moving the electron from the ground state (lowest possible energy) to a higher state requires us to shine light on it. Light of high enough energy, like UV, X-rays, or gamma rays, may move the electron so far away that it is no longer associated with that atom, and this is ionization.

If this happens to an inner-shell electron (one that spends most of its time in or near the nucleus), there is now a vacancy in that shell. One of the other electrons (in a higher shell) can jump down to fill that hole. If that happens, the electron from the upper energy level has to lose some energy to make the transition, so it ejects a photon. When this happens between two energy levels which are close together, this may be a visible-light photon or a UV photon. If the energies of the levels are different enough, an X-ray photon will be ejected. Because the number of possible transitions like this is finite for a given kind of atom, this spectrum is called **discrete**.

The particular makeup of the nucleus and the interactions between orbiting electrons combine to give a different discrete spectrum for each element. For this reason, the X-rays produced in this manner are called **characteristic X-rays**, meaning their energies are characteristic of the atoms whose transitions produced them. The labeling process for characteristic X-rays is confusing, but the shell in which the vacancy is opened is labeled first by a letter (\(K\) = ground state or \(n = 1\) state, \(L\) = first excited state or \(n = 2\) state, \(M\) = second excited state or \(n = 3\) state, etc.) and then either 1) by a Greek letter indicating how many levels the electron filling the vacancy had to jump (\(\alpha = 1\), \(\beta = 2\), etc.), known as Siegbahn notation or 2) by another letter indicating the original location of the newly-moved electron (K-L\(_1\), etc.) known as IUPAC notation.

For hydrogen, the physics involved is relatively simple. We can pretend that the single electron moves in a circle around the single proton (although this is wrong in general for reasons described below, it does work for the case of hydrogen). Anything moving in a circle must be under the influence of a **centripetal** force, which would have a magnitude of \(m_e v^2 / r\) for the electron. For the hydrogen atom, the force is provided by the Coulomb attraction between the proton and electron. We can then write

\[
\frac{m_e v^2}{r} = \frac{k Z_1 Z_2 e^2}{r^2}
\]

where \(r\) is the radius of the circle and the charges on the two particles involved are \(Z_1 \text{ e}\) and \(Z_2 \text{ e}\). For the hydrogen atom, \(Z_1 = Z_2 = 1\). The constant \(k\) is a measure of the strength of the electrostatic attraction between the two particles, and in MKS units is
8.99 \times 10^9 \text{ N m}^2/\text{C}^2. The electron’s classical velocity is represented by \( v \). One problem here is that there are two unknowns in this equation: \( v \) and \( r \). We will need one more equation, and it was first proposed by Niels Bohr (winner of the 1922 Nobel Prize for Physics). He suggested that the electron’s angular momentum (remember that the classical angular momentum for a point particle would be \( L = m v r \)) was quantized, or restricted to certain possible values.

As has happened many times before in physics, the careful exploration of the consequences of one seemingly crazy idea changed the way we understand the world. This wasn’t a complete lightning strike from the blue; Bohr was making a logical extension to another new and weird idea, namely that the electron could act like a wave as well as a particle (de Broglie, Nobel Prize in Physics, 1929). Anyway, the mathematical statement of Bohr’s hypothesis can be written as

\[
L = m vr = \frac{nh}{2\pi}
\]

Again, the \( h \) is Planck’s constant. The letter \( n \) represents any integer greater than zero (that’s the quantized part; if \( n \) could be any real number, we’d be back to classical mechanics. For this reason, \( n \) is known as the principal quantum number.). Substituting this formula into the previous one and solving for the radius of the orbit gives us (after some algebra)

\[
r = \frac{n^2 h^2}{4\pi^2 mkZ e^2}
\]

Notice that we’ve gotten rid of one of the \( Z \)’s and dropped the subscript on the remaining one. This is because what we’re doing would apply to any one-electron atom (i.e., if we have singly-ionized helium, doubly ionized lithium, etc.) The \( Z \) that remains just counts the number of protons in the nucleus. We can connect the radius to the electron’s energy using another classical expression, which is that the energy of a particle bound in a Coulomb potential is

\[
E = \frac{1}{2} m v^2 - \frac{kZ e^2}{r}
\]

which leads directly to

\[
E = -\frac{kZ e^2}{2r} = -\left(\frac{2\pi^2 mk^2 e^4}{h^2}\right) \frac{Z^2}{n^2}
\]

If you work out the constant in parentheses, you’ll find that you can express this as
For hydrogen, \( Z = 1 \) and we get what is hopefully a familiar formula. This will tell us the energy difference between any two states of hydrogen. For example, if we’d like to know how much energy it would take to ionize a hydrogen atom in its ground state, we can look at the difference in the formula above for \( n = \infty \) (corresponding to an electron completely removed from the proton, or ionized) and \( n = 1 \) (the ground state, which represents the lowest energy the atom can have). We’ll get 13.56 eV, meaning a photon that size will ionize hydrogen. Similarly, if a free electron and free proton find each other, they can get together by emitting a single 13.56 eV photon (ultraviolet region of the spectrum) or by emitting several photons of lower energies.

We can also find the energies of other, less extreme transitions. If the electron moves down, from the \( n = 3 \) level to the \( n = 2 \) level, the energy difference is about 1.88 eV, which means we get light with a wavelength of about 656 nm. This is in the red end of the visible spectrum, and is known to astronomers as the **Hydrogen alpha** line.

If we do some checking, it seems that the formula we have above doesn’t work so well for atoms other than hydrogen. For example, for helium (with \( Z = 2 \)) in the ground state, it predicts an ionization energy of \( 2^2 \times 13.56 \) eV, or about 54 eV. The actual ionization energy for helium is about 24.6 eV. The reason for the discrepancy is that the two electrons in neutral helium tend to partially shield each other from the charge of the nucleus. Notice, though, that the second ionization energy for helium ( = energy to remove the second electron after the first has already been removed) is 54.5 eV! As advertised, this works well for one-electron atoms.

Also, although we can’t expect to encounter heavy nuclei surrounded by only one electron under normal circumstances, our formula works reasonably well when describing the behavior of the innermost electrons in non-ionized conditions. For example, if we had a lead (\( Z = 82 \)) nucleus with only one electron, we would expect an ionization energy of about 91 keV. In fact, if one of the two electrons in the lowest energy level of a lead atom is removed and an electron from the outermost levels replaces it, we will get an X-ray photon with an energy of about 88 keV. Not too bad. These high-energy characteristic X-rays can cause problems with other kinds of imaging. We’ll see them again when we look at nuclear medicine.

Also, as you might expect, the classical ideas we used to arrive at the formula above will fall apart if examined closely enough. First, it has been known for a few hundred years that the orbits of particles bound by a force that is proportional to \( 1/r^2 \) (as the Coulomb force and the gravitational force are) are not typically circles, but rather **conic sections** (circles, ellipses, parabolas, and hyperbolas). If the orbit in question is an ellipse (flattened circle), it will take at least one additional number to describe its shape, and this is called the **angular momentum quantum number**, a non-negative integer less than \( n \) which is usually denoted by \( \ell \). In another departure from classical mechanics, the electron can actually have **zero** orbital angular momentum (\( \ell = 0 \)).
There is also a magnetic quantum number $m_l$ which can take on any integer value from $-\ell$ to $\ell$. Two electrons in orbits with the same values of $n$ and $\ell$ but different values of $m_l$ will have the same energies except in the presence of a magnetic field. The electron has two more quantum numbers, $s$ and $m_s$. For all electrons everywhere, $s = \frac{1}{2}$. This is known as the electron’s intrinsic angular momentum, or spin. The relationship between $s$ and $m_s$ is similar to that between $\ell$ and $m_l$: $m_s$ can take on values between $-s$ and $s$, which in the case of an electron means two choices, usually referred to as up or down.

These facts and the Pauli exclusion principle can make clear the energy-shell concept from chemistry. For the $n = 1$ state, $\ell = 0$ so $m_l = 0$ also. All electrons have $s = \frac{1}{2}$, so we can only put two electrons in the $n = 1$ state because we have two choices for $m_s$. If we look at the $n = 2$ state, we still have an $\ell = 0$ option (giving us two more places to put electrons, one for each value of $m_s$), but we can also have $\ell = 1$. That means we can have $m_l = -1$, 0, or 1. Again, $m_s$ doubles the possibilities for each of these, so we get 2 openings for $n = 2$, $\ell = 0$ and 6 for $n = 2$, $\ell = 1$ for a total of 8 electrons in the second shell.

Energy level transitions are not the only way X-rays are produced. When electrons moving at high speeds approach a nucleus, they tend to be decelerated by the Coulomb interaction (attraction/repulsion of unlike/like charges). Accelerated charges emit electromagnetic radiation (as a side note, if electrons really moved in circles around the nucleus like planets around the sun, they would be constantly accelerated, would constantly radiate energy, and would fall into the nucleus in far less than one second. The fact that this obviously does not happen was a glaring failure of classical mechanics to explain the microscopic world, and pointed out the need for what became quantum mechanics). The probability that an electron will emit radiation in this situation is known as the cross section (symbol $\sigma$) for X-ray production. If we were trying to calculate the chance of two spheres colliding, one of the important things to know would be the cross-sectional area of each; it’s much easier to hit one basketball with another than it is to hit one BB with another. We can derive (not easily) the formula below

$$\sigma \propto \frac{z^4 Z^2}{137} \left( \frac{e^2}{mc^2} \right)^2$$

In this formula, $z$ is the number of elementary charges on the projectile, $Z$ is the number in the target nucleus, and $m$ is the mass of the projectile. This leads us to ask if the electron is really the best particle to use for X-ray production, since it has only a single charge (so $z = 1$). We could try using something like an alpha particle, which is really just the nucleus of a helium atom (2 protons + 2 neutrons). The $z$ value would then be 2, so the cross section would go up by a factor of $2^4$ or 16. Of course, the mass of an alpha particle is about 7500 times the mass of an electron, so that would drive the cross section down by $(1/7500)^2$. We’d better stick to electrons.
This kind of radiation is known as **Bremsstrahlung**, which is German for "braking radiation". The Bremsstrahlung spectrum is continuous, meaning that if we have enough of these electrons interacting with nuclei, we will get photons of all wavelengths in a range. The limit to this range is reached when the electron gives up all of its kinetic energy in a single interaction. For example, if an electron is accelerated through a potential difference of 120,000 volts, so that it has a kinetic energy of 120 keV, it could produce a single X-ray photon with that much energy or multiple photons with smaller energies. It can’t, of course, give off photons with energies higher than 120 keV. This gives the short-wavelength cutoff to the Bremsstrahlung spectrum.

In the image above, taken from http://www.amptek.com/eclipse_3.png, we can see the characteristic lines from the X-ray target (labeled as Ag Kα and AgKβ and located at approximately 22.2 keV and 25.0 keV). The rest of the "hill" is the continuous spectrum produced by the Bremsstrahlung radiation. The spectrum drops off to nearly zero at 30 keV, as we might expect with electrons moving across a 30 kV potential difference. Because the detector is not perfect and has a finite energy resolution, we don’t see the razor-sharp cutoff we might otherwise expect.

On the other end of the spectrum, notice that there is a sharp cutoff at a little less than 1 keV. This wasn’t predicted in our previous analysis, but it’s because of another practical consideration: the X-ray detector is behind a Beryllium window. This allows a vacuum to be created around the detector for improved performance. Of course, the window will provide a (very small) amount of shielding for the detector. The lowest-
energy photons will not be able to penetrate the window and reach the detector, but that’s fine.

The removal of long wavelengths from the X-ray spectrum is known as beam hardening because the X-rays that make it through the obstruction will have a higher average energy after leaving the lower-energy X-rays behind. Low energy X-rays are sometimes referred to as “soft” X-rays, in contrast to higher-energy “hard” X-rays. This is not necessarily a problem, depending on the degree of beam hardening. The lowest-energy photons would have only a vanishingly small chance of making it through the body anyway, which means they would add to the absorbed dose of radiation without adding anything to the imaging process.

For this reason, the X-ray beam may be passed through some thickness of aluminum (known as a filter) to remove the undesirable low-energy photons. Remember that altering the potential difference (voltage) between the anode and cathode will change the maximum energy of the beam, but filtration is the only way to change the minimum energy, which would otherwise be approximately zero.

Production of X-rays

As you may have gathered by now, X-rays are produced when free electrons are accelerated across a large potential difference and made to collide with a target which has a high atomic number. A low atomic number would make Bremsstrahlung much less likely and would also give characteristic “X-rays” that would tend to be in the ultraviolet or soft X-ray region of the spectrum.

The electrons are freed by being boiled off of a hot filament, or cathode. Once free, they can respond to the large electric field produced by the X-ray machine’s high-voltage power supply and rapidly accelerate towards the anode, or positive terminal. This has to take place in a vacuum because the electron would otherwise lose most of its energy in collisions with air molecules (a microscopic version of air resistance).

When the electron slams into the anode (or a target placed between it and the filament), it will generate X-rays by both methods discussed above. Of course, not all of the kinetic energy gained by the electrons is successfully converted to X-ray photons. For electrons, the ratio of kinetic energy going to the production of X-rays via the Bremsstrahlung process to the energy which goes into ionizing the target is

\[
\frac{\text{Bremsstrahlung}}{\text{ionization}} \approx \frac{KE_e}{700} (Z + 1.2)
\]

where the electron’s kinetic energy is given in MeV. For diagnostic energies (we can use 150 keV = 0.15 MeV as an upper limit) and a target with Z = 74, we get around

\[\text{p.205, An Introduction to Nuclear Physics, Cottingham & Greenwood, Cambridge University Press}\]
1.6%, meaning >98% of the energy ultimately goes into heating the target. The problem is worse at lower beam energies.

This presents unique challenges to the designer of an X-ray tube. First, we want a material with a large value of Z. We definitely do not want a target which is itself radioactive, since that will only complicate the output spectrum. This already restricts us to Z = 72 to Z = 83. The solid, stable materials near the bottom of the periodic table tend to be quite expensive (platinum, iridium, osmium, gold, etc.), although that’s not the largest drawback for many of them.

There are a few cheap choices, such as bismuth, lead, and tungsten. Our next requirement is that the target needs to be able to do something with the very large amounts of heat dumped into it by the electron beam. We need decent thermal conductivity so that the heat can be removed from the target, we’d like a large specific heat so that its temperature does not climb quickly from the absorption of energy, and we’d also like a high melting point for obvious reasons.

Let’s look at the candidates:

<table>
<thead>
<tr>
<th>Element</th>
<th>Z</th>
<th>Thermal Cond. (W/K-m)</th>
<th>Sp. Heat (J K/g)</th>
<th>Melting Pt (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hafnium</td>
<td>72</td>
<td>23</td>
<td>0.144</td>
<td>2503</td>
</tr>
<tr>
<td>Tantalum</td>
<td>73</td>
<td>57.5</td>
<td>0.14</td>
<td>3269</td>
</tr>
<tr>
<td>Tungsten</td>
<td>74</td>
<td>174</td>
<td>0.132</td>
<td>3680</td>
</tr>
<tr>
<td>Rhenium</td>
<td>75</td>
<td>47.9</td>
<td>0.137</td>
<td>3453</td>
</tr>
<tr>
<td>Osmium</td>
<td>76</td>
<td>87.6</td>
<td>0.13</td>
<td>3327</td>
</tr>
<tr>
<td>Iridium</td>
<td>77</td>
<td>147</td>
<td>0.131</td>
<td>2683</td>
</tr>
<tr>
<td>Platinum</td>
<td>78</td>
<td>71.6</td>
<td>0.133</td>
<td>2045</td>
</tr>
<tr>
<td>Gold</td>
<td>79</td>
<td>317</td>
<td>0.129</td>
<td>1338</td>
</tr>
<tr>
<td>Mercury</td>
<td>80</td>
<td>8.34</td>
<td>0.14</td>
<td>234</td>
</tr>
<tr>
<td>Thallium</td>
<td>81</td>
<td>46.1</td>
<td>0.129</td>
<td>577</td>
</tr>
<tr>
<td>Lead</td>
<td>82</td>
<td>35.3</td>
<td>0.129</td>
<td>601</td>
</tr>
<tr>
<td>Bismuth</td>
<td>83</td>
<td>7.87</td>
<td>0.122</td>
<td>545</td>
</tr>
</tbody>
</table>

As we can see, the specific heats are all about the same. The thermal conductivities show a much larger range of values, though, with only gold, iridium, and tungsten having $\kappa > 100$ W/K m. Looking at melting points would rule out the lower part of the table. After examining this data, it probably won’t surprise you that the targets in X-ray machines are typically made from tungsten. It’s just about the ideal material for this purpose (also, at under $50$/kg, it’s relatively cheap).

A sharp image will be produced if the focal spot (the part of the target hit by the electron beam) is small. Unfortunately, if we direct the electron beam at a tiny region, we can burn a hole through the target very quickly. To balance the need for sharp images with the need to prolong the life of the X-ray tube, two methods are commonly used. First, we can tilt the target as shown below:
We won’t get into the details of the directional distribution of Bremsstrahlung, but at the (relatively low) energies found in X-ray tubes, the photons are preferentially emitted at a 90° angle to the electron’s travel. For image production purposes, the focal spot will be very small, but the area of the target over which the electron beam is spread is much larger. This keeps heat from building up too high in one place. Also, the target is really part of a larger rotating anode, which further spreads the heat:

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When combined with liquid cooling, the heat issues can be effectively managed.

**Brightness, Contrast, and Resolution**

What kind of picture might we get if we used very high energy X-rays? Larger energies mean greater penetrating power. The maximum in a medical X-ray machine is usually around 120 keV. Looking back to the first X-ray, remember that Röntgen saw bones in his hand; that tells us that there must be a difference in the way X-rays interact with your bones as compared to muscles/other tissue.

This difference gives us **contrast**. The set of pictures below, from [http://en.wikipedia.org/wiki/Contrast_%28vision%29](http://en.wikipedia.org/wiki/Contrast_%28vision%29), shows a gradual change in the contrast of the image.
The image on the bottom left is a low-contrast representation. As you move clockwise through the pictures, the contrast increases. As contrast decreases, it becomes harder to distinguish different features from each other (the sky tends to blend with the ocean, etc.). This is one of the things to watch for when buying a TV – you want the black part of the screen to be as black as possible. If “black” is really just gray, the picture will not look as vivid.

Another important factor in image quality is resolution. In medical imaging, this is usually measured in **line pairs per inch** (or line pairs per millimeter outside the US). This tells you how close two lines can be to one another and still be distinguished as two separate lines. (Depending on the application, the resolution can be several line pairs per millimeter to 20 or more – keep in mind that a millimeter is about the thickness of a dime!). In the images from the Hubble telescope below, the first is a low-resolution picture while the second is high resolution. The third is the result of magnifying the low-resolution image. Notice that the picture is larger, but it is blurry in comparison to the other image. That’s because the information present in the high-res picture is absent in the magnified low-res picture. This is an interesting effect to watch for in the movies and on TV. It’s very common for someone in the “computer lab” to take a grainy photo captured on someone’s cell phone at a crime scene and blow it up while sharpening it to some ridiculous level so that the billboard across the street reflected in the victim’s eyes is clearly readable. It doesn’t happen that way in real life.
Images from http://hubblesite.org/gallery/album/solar_system_collection/pr2005029a/ - Left image is low resolution, lower image is high resolution.
Brightness is more of a measure of how many photons actually hit the X-ray film or detector. In general, we might think that brighter is better, but it’s worth remembering that there are limits. You probably already know that it’s a bad idea to take a picture of someone who has his/her back to the Sun. The reason is that the Sun is far brighter than the person’s face (which is necessarily in the shadows if facing away from the Sun). A quick exposure time which keeps the Sun from overwhelming everything will not be long enough to gather the relatively few photons from the person’s shaded face. The film only has a certain dynamic range, or difference between the most and least...
exposed parts of it. A CCD, which is the electronic equivalent of film in a digital camera, typically has a larger dynamic range than film. This is one of the reasons it is preferred over film by astronomers when they are capturing images from large telescopes. Your eyes have a large dynamic range overall (the Sun has an apparent brightness about 10 trillion times greater than the dimmest stars you can see with the naked eye), although the full range is not available at any one time. In other words, if you go from a bright summer day outside to a dimly lit room, you’re going to trip over things for a minute or two until your eyes adjust; similarly, the bright lights of a car behind you at night are far more annoying than the same lights would be in the daytime.

Below are a few versions of the picture of the Moon which have had their contrast and/or brightness artificially altered. You can see that there are problems with too much/too little of anything (contrast or brightness).

<table>
<thead>
<tr>
<th></th>
<th>Low brightness</th>
<th>Medium brightness</th>
<th>High brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Contrast</td>
<td><img src="image" alt="Low Contrast Image" /></td>
<td><img src="image" alt="Medium Contrast Image" /></td>
<td><img src="image" alt="High Contrast Image" /></td>
</tr>
<tr>
<td>Medium Contrast</td>
<td><img src="image" alt="Low Contrast Image" /></td>
<td><img src="image" alt="Medium Contrast Image" /></td>
<td><img src="image" alt="High Contrast Image" /></td>
</tr>
<tr>
<td>High Contrast</td>
<td><img src="image" alt="Low Contrast Image" /></td>
<td><img src="image" alt="Medium Contrast Image" /></td>
<td><img src="image" alt="High Contrast Image" /></td>
</tr>
</tbody>
</table>

So, what does this have to do with X-rays? Well, we know that if the beam’s energy (the maximum of which is the potential difference between the anode & cathode in the X-ray tube) is too low, the X-rays won’t penetrate the patient and will just give a radiation dose for no purpose (we’ll get an image with low brightness). On the other hand, if the beam energy is too high for the part of the body being imaged, there will be very little contrast between the different structures in the body. Very high energy X-rays will go through bone approximately as easily as they go through other tissue, and the resulting image will have very high brightness and very low contrast, which will look washed out.

The brightness of the exposure depends on the number of photons hitting the film, which depends in turn on the rate of electrons produced (= current through the filament) multiplied by the exposure time.
Attenuation & Scatter

What are we really measuring with an X-ray? Some intensity of radiation goes into the patient, and a lesser intensity of it comes out of the patient and interacts with the film or CCD. All we’re really doing is comparing output to input. We say that the beam is **attenuated**, or reduced in intensity, by passing through the patient. The amount of attenuation depends on the thickness of the material and its composition. As a general rule, “high Z” materials (those with a large number of protons, like lead) are better at stopping X-rays than low Z materials (like water). This is because elements/compounds with lots of protons also have lots of electrons to make them electrically neutral, and the incoming radiation is more likely to interact with the electrons than anything else. Roughly, the attenuation goes as $Z^3$, meaning lead ($Z = 82$) will be $(82/26)^3 = \approx 30 \times$ better at stopping X-rays than iron ($Z = 26$).

A general result is that the intensity $I$ changes with distance $x$ as

$$\frac{\Delta I}{\Delta x} = -\mu I$$

so that the change in intensity depends on the intensity itself (which makes sense; we would expect these interactions with matter to remove a percentage of the beam every centimeter rather than a fixed number of photons per centimeter). The ability of a material to block these X-rays is called the **attenuation coefficient**, and is represented by $\mu$. We can integrate this equation and rewrite it as

$$I(x) = I_0 e^{-\mu x}$$

where $I_0$ is the intensity of the beam at the point where $x = 0$. In one sense, this is a simple problem. We emit a known intensity of radiation and we receive a different intensity. We can (if necessary) measure the “thickness” of the patient along a line between the source and detector and quickly calculate $\mu$. A high value of $\mu$ in a medical X-ray would indicate bone, or some kind of surgical appliance such as a stainless steel or titanium hip joint, etc. A low value of $\mu$ could mean fat or muscle, while a very low value for $\mu$ would indicate something like air (in the lungs).

Of course, there are some complications. First of all, people are not segmented neatly. Along a given path between the X-ray tube and the film, the beam may well encounter skin, muscle, fat, bone, and air, all in a single line. The attenuation will then be a function of all of these things, so that we’ll get something more like

$$I(x) = I_0 e^{-\left(\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4 + \ldots\right)}$$
where $\mu_1$ might be the attenuation coefficient for bone and $x_1$ the thickness of bone traversed, and $\mu_2$ might be the attenuation coefficient for fat and $x_2$ the thickness of fat, etc. In the most general case, we could imagine breaking the path between the X-ray tube and the film into millions of microscopic little distance elements $x_i$, each with its own value of $\mu_i$. We would then add up all of the $\mu_i x_i$ terms and put them in the exponential. This leads to the idea of integration of a function from calculus. We would replace the many $\mu_i$ terms by the function $\mu(x)$ and take the integral of that over the path through the body.

If the area to be examined has an attenuation coefficient that would otherwise be too low to make an X-ray useful, we can sometimes add a contrast agent. This is a higher-Z material (frequently barium sulfate) that is much better at stopping X-rays than soft tissue. Below (from http://web.indstate.edu/thcm/e/micro/anaercult/img023.JPG) we see the results of a scan searching for diverticulitis.

![Diverticula: account for >350,000 cases of peritonitis/yr](http://web.indstate.edu/thcm/e/micro/anaercult/img023.JPG)

One problem we will have to resolve is the fact that this method (a regular planar X-ray, like you might have taken if you think you’ve broken a bone) is really only able to measure the ratio $I / I_0$. That gives us information about the total product of all of the $\mu_i x_i$ terms, but tells us nothing about their relative sizes. In other words, if the X-ray film has only been lightly exposed in a region, is that because there was a relatively thin layer of bone in the way, or a very large layer of fat, or a small gold necklace, or what? The way around this will lead (later) to the computed tomography, or CT scan (FYI, people in the medical field typically do not refer to “CAT” scans anymore).

A final difficulty to keep in mind is that, in addition to the dependence of $\mu$ on the material it encounters, there is also a dependence on the energy of the X-ray. For a few examples, see the table below.
<table>
<thead>
<tr>
<th>Material</th>
<th>50 keV</th>
<th>100 keV</th>
<th>1000 keV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>0.00024</td>
<td>0.000184</td>
<td>0.000076</td>
</tr>
<tr>
<td>Water</td>
<td>0.2261</td>
<td>0.1706</td>
<td>0.0707</td>
</tr>
<tr>
<td>Bone</td>
<td>0.7750</td>
<td>0.3411</td>
<td>0.1214</td>
</tr>
<tr>
<td>Fat</td>
<td>0.1911</td>
<td>0.1554</td>
<td>0.0655</td>
</tr>
<tr>
<td>Lead</td>
<td>89.529</td>
<td>62.722</td>
<td>0.7973</td>
</tr>
</tbody>
</table>

The values of $\mu$ above are measured in cm$^{-1}$. For example, a 5 cm layer of lead could be expected to block about 98% of 1000 keV photons hitting it, from the math below:

$$\ e^{-5.0*0.7973} = 0.0186 \cong 2\%$$

If about 2% of the photons get through, the remaining 98% are blocked.

What physical processes are removing these X-rays from the beam? This involves the details of the interaction between electromagnetic radiation and matter. The theory which describes this is known as quantum electrodynamics, and is the most successful scientific theory ever. Prediction and experiment agree to better than 1 part in $10^{10}$! This theory is very complicated, but we will be looking at a subset of cases that are useful in medical imaging applications.

The primary interactions we will examine will be the **photoelectric** interaction and **Compton scattering**. In each of these cases, the photon will lose some or all of its energy. For this reason, these are inelastic scattering processes, meaning (just as in the case with the classical physics of collisions) kinetic energy is transformed into some other kind of energy and is not, by itself, conserved.

**The Photoelectric Effect**

The photoelectric effect was first explained by Einstein. It had already been observed that certain colors of light had the ability to liberate electrons from the metal to which they were bound. Treating the incoming light as a wave predicted things that were not observed experimentally. For example, a light wave of any color (= frequency) should be able to shake the electron back and forth until it is freed as long as its amplitude (size of motion) is large enough. Also, there should be some time delay between the wave first moving the electron and when it is able to finally shake it free. Neither of these predictions are true, and this paved the way for the description of light as a particle (photon).

We now realize that the photon has to be carrying enough energy to free the bound electron, and that the energy of a particular photon only depends on its frequency. Ultraviolet light is able to do this easily, and infrared light can't do it at all.

The amplitude of a classical wave would correspond to the number of photons. Since there is only an infinitesimal chance of more than one photon interacting with an electron at a time (before the invention of lasers), the number of photons really only becomes important if the frequency is large enough for each to remove an electron.
Similarly, if you can’t throw a rock hard enough to penetrate a tank’s armor, having 500 friends help you throw rocks at the same speed will also not work. The rock is only effective if it carries more energy in a single package.

While visible photons can interact with the conduction electrons in a metal, high energy photons such as X-rays are more likely to interact with the more tightly-bound electrons in the inner shells of the atoms. Again, there is a minimum energy the photon must have for this interaction to occur; if we’re talking about the electrons in the lowest energy level of lead, that cutoff is about 88 keV. X-rays with energies lower than this will be unable to eject electrons from the innermost shell, but they can work on the less-tightly bound shells of higher levels. If we watch the results as the energy of the X-rays is increased, we will see a sudden spike at 88 keV as two more electrons in each of the many lead atoms are now available to participate in the interaction.

As an approximation for the energy range most commonly seen in medical imaging, we can say that the cross section for photoelectric interactions is proportional to \( Z^4 / E^3 \) where \( E \) is the energy of the photon and \( Z \) is the atomic number of the atom involved. This formula does not apply if the energy is at or above a characteristic X-ray energy (i.e., around 88 keV in lead).

In the photoelectric interaction, the photon is absorbed and therefore disappears. Almost all of the energy goes into the electron. The remainder goes into the atom from which the electron is ejected, and the reason is that momentum must be conserved. The photon has an energy given by \( E = h f \) and a momentum given by \( E = h / \lambda \). Both momentum and energy must be conserved in the interaction, and that can only happen if the atom carries away some momentum. Of course, the atom has to have kinetic energy if it has momentum, and we can write the connection between the two as

\[
KE = \frac{p^2}{2m_{\text{atom}}}
\]

Since the atom’s mass is so large compared to the electron’s mass (many thousands of times larger), it only takes a tiny amount of energy away. The rest of the energy is available to the electron. After overcoming its binding energy (= paying off its debt of “negative energy” to the atom), the remainder of the energy becomes KE for the electron. As we’ll see later, electrons have a very short range in matter. This means the kinetic energy will be deposited in a small volume. This energy therefore contributes to the radiation dose to the patient, but not to the imaging process itself.

What happens to the newly-ionized atom? Now that it has a vacancy in the very desirable inner shell, it is likely that one of the outer-shell electrons will fall down to fill it. This of course will require the emission of a photon to rid the outer-shell electron of its excess energy. The photon’s energy will be \( |E_{\text{inner shell}} - E_{\text{outer shell}}| \), but since the outer shell energy is usually a few eV while the inner shell energy may be tens of keV, we can very effectively approximate the photon’s energy as just \( |E_{\text{inner shell}}| \). This new X-ray emission is known as fluorescence. This is not something that’s particularly
desirable in the imaging process, because knowing the original source of the radiation is a big part of the problem. If you’re using an X-ray beam with a maximum energy of 120 keV, it will also contain all energies down to the beam filtration limit (maybe 40 keV, for example). How can you tell if the film has captured an 88 keV X-ray from the original beam or if it’s a fluorescence X-ray from the lead apron protecting the patient? You can’t.

Instead of emitting a single 88 keV X-ray (again, we’re assuming the inner shell of lead is responsible for this X-ray. In reality, there will be photons from different levels of all the materials in the beam’s path, from the iron in the table to the calcium in the patient’s bones), there might be multiple lower energy photons. An electron might drop from the n=2 shell to the n=1 shell, emitting a photon and leaving a vacancy in the n=2 shell that would then be filled by an electron in a higher shell, which would also have to emit a photon, etc.

Also, instead of having the remaining energy carried off by one or more photons, we could have it carried off by an Auger electron. Imagine that the X-ray produced hits an outer shell electron in the same atom. This electron will be ejected (again by the photoelectric effect) just as the original electron was.

**Compton Scattering**

What will happen if the incoming X-ray photon hits a free electron instead of a bound electron? In this case, the photon cannot be absorbed by the electron. Particles with mass have a different relationship between their energies and momenta than the one that applies to photons. More simply, there is no way to conserve both energy and momentum in the reaction below:

$$\gamma + e^- \rightarrow e^- \quad \text{not allowed}$$

This doesn’t mean the two particles can’t interact, though. What we will get in this case will be

$$\gamma + e^- \rightarrow \gamma + e^-$$

where the two particles will exchange energy and momentum during the interaction. This process is known as **Compton scattering**. Because the only two particles involved in Compton scattering are the photon and the electron, and because the energy-momentum relationship of each is well known, we can relate the change in the photon’s direction to its initial and final energies. The derivation of the result is not overly difficult, but it does involve some reasonably tedious algebra, so we will just quote it directly:
\[ \Delta \lambda = \frac{\hbar}{m_e c} (1 - \cos \theta) \]

The change in the photon’s wavelength (which can give us the energy change very quickly) is \( \Delta \lambda \). On dimensional grounds, therefore, \( \hbar/(m_e c) \) must be a wavelength as well, and it is known as the **Compton wavelength of the electron**, which is \( 2.43 \times 10^{-12} \) m. This tells us that a small change in the direction of the photon necessarily means a small change in its energy. Conversely, the largest possible change in the photon’s energy occurs when the scattering angle is equal to 180°. This is known as **backscattering** since the photon heads back along its initial direction. For example, if a 100 keV photon is backscattered, its energy after scattering will be 71.893 keV. The plot below shows the energy of a scattered 100 keV photon as a function of angle.

In X-ray procedures, the photoelectric interaction and Compton scattering are the most important interactions between light and matter. We know that the photoelectric scattering probability decreases roughly as \( 1/E^3 \), and experiment shows that the Compton scattering probability also decreases with increasing photon energy, but a third process appears at higher energies.

**Pair Production**

When the energy of an incoming photon is greater than about 1 MeV (one million electron volts), the photon can interact with a bound electron (not free – energy and momentum still must be conserved) to produce an electron and a positron, which is the anti-matter version of an electron. When the positron meets up with another electron, which happens quickly, the two will annihilate each other, typically producing two
photons. The cross section for this process is proportional to $Z^2$, and it increases (in a more complicated way) with energy, in contrast to the other two types of interaction.

Because this process is not possible below the 1 MeV threshold, it is not a concern in medical imaging.

**Effects and Reduction of Scatter**

Measuring the attenuation coefficient (which is all we’re really doing in a planar X-ray) relies on the assumption that the X-rays are traveling in a straight line from the tube to their final position on the film or CCD. As we saw in the last section, photons are very likely to change direction after interacting with matter. For an extreme case, imagine the diagram below depicts the imaging process for locating a piece of lead in a body.

On the left we have the “ideal” case, where there is no scattering and the lead sphere blocks all of the X-rays incident on it, leaving a clean shadow on the film. On the right, we have scattering events in the body which happen to redirect X-rays to what would be the dark spot on the film, exposing it more than in the case on the left.

This is an idealized case of the general result, which is that scatter reduces image contrast by adding illumination to the background essentially everywhere. A common measure of this effect is the **signal to noise ratio**, or SNR.

Each graph below shows half a period of a sine wave with two different values of random noise superimposed on it. The signal is the same strength in each graph, but the noise floor is larger in the right graph. It’s much harder to see the signal in the plot on the right.
There are some ways to reduce the effects of scatter. First, we can **collimate**, or directionally restrict, the beam before it enters the patient. This may be done by lead shutters placed between the patient and the X-ray tube. It's important to note that there is no way to **focus** X-rays because their wavelengths are so small. The shutters function like lamp shades, keeping radiation from going into areas where we know we **don't** want it by absorbing it (i.e., if you're getting an X-ray to see if your hand is broken, there is no need to irradiate your entire arm/upper body).

A similar device, known as a **grid**, can be placed between the patient and the film or detector. It consists of a pattern of good and poor X-ray absorbers arranged so that X-rays traveling along a straight line from the tube to the film are allowed and those that move along different paths (because they've been scattered) are rejected. In the picture below, the black lines above the film represent the absorbing parts of the grid. As you can see, they will allow the X-ray moving in a straight line (green dotted line) to hit the film while the X-ray which has been scattered and had its direction changed (red dashed lines) is absorbed by the lead in the grid.
This is one of the big benefits X-ray and CT scans have as compared to planar nuclear medicine and SPECT scans: the ability to "aim" the radiation in the beginning and to only look for radiation in that direction afterwards is very helpful.

**Nuclear Physics**

X-rays and CT scans involve the *transmission* of radiation from one side of the body to the other. As a general rule, they provide information about *structure* rather than *function*. For example, the mammogram commonly given for breast cancer screening typically has a resolution in the tens of microns, meaning it can accurately locate very tiny spots where the attenuation coefficient is higher or lower than its surroundings. Unfortunately, it can’t tell whether that tiny spot is a harmless calcification or the initial stages of a tumor. That requires information about function, and one of the most common and oldest ways to discover that is to bind a radioactive substance (a tracer) to some kind of biologically active compound and then follow the distribution of the tracer through the body. To understand how this works, we need some background information about radioactive decay.

Radioactive decay is a *nuclear* process rather than an atomic process. This decay occurs when a nucleus is somehow out of balance; there are too many protons, or neutrons, or both, or there is too much energy in the nucleus. The three most common types of radioactive decay particles are *alpha*, *beta*, and *gamma*. The α particle consists of two protons and two neutrons which are bound together very tightly. This can also be thought of as the nucleus of a helium atom, or He²⁺. The β particle is most commonly thought of as an electron, although the positron is also considered a β particle (it’s generally a good idea to remove all ambiguity by referring to the electron either by name or with the symbols e⁻ or β⁻ and the positron either by name or with the symbols e⁺ or β⁺). Finally, the γ is just a high-energy photon. We'll see later that, while γ rays are typically thought of as more energetic than X-rays, the only real distinction is
in the point of origin; if the photon is produced via electron shell transitions or the Bremsstrahlung process, it is called an X-ray. The photon is called a gamma ray if it comes from nuclear processes (which we'll discuss soon) or the annihilation of matter and antimatter.

The nucleus of an atom is composed of protons and neutrons. The number of protons (usually denoted by $Z$) determines the elemental identity, or atomic number, of the nucleus. The number of neutrons ($N$) has a strong effect on the stability of the nucleus, and adds to its atomic weight, but not to its charge. The total number of nucleons in the nucleus ($A$) is just the sum of the number of neutrons and protons: $A = N + Z$. It's common to write the full identity of the element as

$$^A_z \text{Sy}$$

where $\text{Sy}$ represents the chemical symbol for that particular element. Of course, this is also a little redundant, since knowing either $Z$ or the chemical symbol automatically gives you the other piece of information. For this reason, the $Z$ subscript is sometimes dropped.

If you study the periodic table, you'll notice that elements near the top tend to have weights that are about twice their atomic numbers. This tells us that the number of protons and neutrons is equal. This makes a stable nucleus, at least for small nuclei. If two nuclei have the same value of $Z$ (same element) but different values of $A$, they must necessarily have different numbers of neutrons. These are known as isotopes of a particular element. For example, hydrogen can be $^1\text{H}$ (protium – the common stuff), $^2\text{H}$ (deuterium), or $^3\text{H}$ (tritium). Each has one proton, and they have zero, one, or two neutrons. (It's worth mentioning that the atomic weight is not usually an integer for a few reasons, one being that it is an average of $A$ weighted by abundance of the different isotopes).

Nuclei with the same number of neutrons but different numbers of protons are known as isotones. Examples of this would be $^{18}\text{O}$ and $^{20}\text{Ne}$, each having 10 neutrons. If two nuclei have the same total number of nucleons, they are isobars (this name comes from the fact that both neutrons and protons are classified as baryons by particle physicists). $^{18}\text{O}$ and $^{18}\text{F}$ are isobars.

\textbf{α Decay}

The α decay process is reasonably simple. Notice that the rough formula of $A = 2Z$ which works reasonably well at the top of the periodic table is not followed near the bottom of the table at all. We see that there are more neutrons than protons in larger nuclei. To understand the reason, we need to look more deeply into the forces at work.

There are thought to be only four fundamental forces in the universe: gravity and electromagnetism, which you've already seen in introductory physics, and the strong and weak nuclear forces, which are probably new to you (in fact, one of the important
Developments of late 20th century physics was the unification of the electromagnetic and weak forces, so that we could more accurately say there are three fundamental forces. Many of the greatest minds since Einstein have devoted their careers to the effort to understand all three forces as different aspects of the same unified force, just as physicists understood a century ago that magnetism and electricity were really a single force).

It's easy enough to understand why the electrons in an atom stick around – they're negatively charged and the nucleus is positively charged, so there is an attraction between the two that we can describe with Coulomb's law. Why, though, do the protons all stick together in the nucleus, where they are much closer to each other than the electrons are to each other? The magnitude of the charge on the proton is the same as that on the electron, but where the electrons are confined to a volume which is about $10^{-10}$ m in diameter, the protons are bunched into a region approximately $10^{-14}$ m in diameter. Since the Coulomb force increases as $1/r^2$, this means the repulsive force between protons is far, far greater than the attractive force between protons and electrons.

This is where the strong nuclear force comes in. It's called the strong force because (try to guess!) it's the strongest of the four forces, roughly 100 times the strength of the electromagnetic force. (Gravity, by the way, is ridiculously weak in comparison – it's around $10^{-38}$ the strength of the strong force!). You may have considered the pieces of the atom to be fundamental particles, but only the electron is. The proton and neutron are actually made of three quarks each. Quarks are currently believed to be fundamental, in that there appear to be no smaller particles “inside” them. Also, quarks have fractional charges that are multiples of one third of the electron charge or $e/3$. We haven't switched to the use of $e/3$ as the fundamental unit of charge because isolated quarks have never been seen. They are always found in groups of two or three, and those groups have charges which are integer multiples of $e$. There are 6 different species of quark (called flavors), known by the names up, down, charm, strange, top, and bottom or the letters u, d, c, s, t, and b. All “normal matter” (the stuff that doesn't disappear in a flash in far less than a nanosecond) is made of up and down quarks. The proton is two ups and a down (uud) while the neutron is two downs and an up (udd). If the charge on an up quark is $+2e/3$, figure out the charge on a down quark.

Anyway, quarks feel the strong force but electrons don't. It's because of another property quarks have called color. This has nothing to do with color as we know it; the name arose because combining three colors of light will give white light, which we consider colorless. The particles we can observe are all "colorless", which is why the quarks combine in threes. The three colors used are red, green and blue. This means that, to completely describe a quark, we could say it's a "red up" or a "green down". It's the strong force that binds the three quarks into protons or neutrons. The strong force is also sometimes known as the color force since it's really acting between the quarks. Electrons have no color, so they don't feel it.

The force between colorless protons and/or neutrons is really a result of the “leftovers” of the color force acting inside the protons and neutrons. This is similar to the way the
van der Waals force between neutral molecules can nevertheless hold them together. The thing that makes the strong and weak forces so different from the other two that you already know about is the range of the interaction. The electromagnetic and gravitational interactions are infinite in range, meaning that there is a force between your toenail and the Andromeda galaxy. The strength of the interaction falls off as $1/r^2$, so the toenail-galaxy force is too ridiculously small to worry about, but it's there, and the same goes for the electrostatic repulsion between an electron in your toenail and one in the Andromeda galaxy.

The strong force has a range of about $10^{-15}$ m, which is about the width of a proton or neutron. In other words, the proton may be attracted to its neighbor by the strong force, but it's not really attracted to the proton or neutron on the other side of a mid-sized nucleus at all. This is where the balance comes in. If the strong force had an infinite range, it would be energetically favorable to dump all the protons and neutrons in the universe into one super-nucleus (although that would make things kind of hard for us).

Instead, the strong force allows neighbors to pull on each other, but the electromagnetic force makes protons repel each other. Although the proton is attracted only to its neighbors, it is repelled by every other proton in the nucleus. Eventually, when the nucleus gets large enough, the repulsion wins and the nucleus will be unstable and want to break into smaller pieces. To postpone this, extra neutrons can be added as “spacers”. The protons are attracted to the neutrons by the strong force, but adding neutrons increases the distance between protons, which decreases the strength of the $1/r^2$ repulsive force. For this reason, one of the largest stable elements, $^{209}$Bi, has 83 protons but 126 neutrons. We've gone from a 1:1 ratio to 3:2.

This trick can't go on forever, though. It turns out that the ratio can be too far out of balance and this can cause its own instabilities. A very useful way for the nucleus to head down to a smaller and more manageable size is for it to eject an $\alpha$ particle. By doing this, we get rid of two protons, but only two neutrons go with them. Because this changes things from $A$ to $A-4$ and $Z$ to $Z-2$, this makes our overall neutron:proton ratio increase, which reduces the repulsive energy in the nucleus. This is a very common way for a large nucleus to decay. We could represent the $\alpha$ decay of $^{238}$U as

$$^{238}U \rightarrow ^{234}Th + \alpha$$

In smoke detectors, the escaping $\alpha$ particles from the decay of $^{241}$Am produce a current which is sensed by electronics in the detector. The $\alpha$ particles are very easy to stop, and even a light density of smoke will do it. When the current is interrupted, the alarm sounds.

This is one of the reasons that $\alpha$ particles are not useful in imaging applications. These particles are relatively heavy (approximately 8000 times the mass of an electron) and therefore even those with the highest decay energies of around 10 MeV are moving relatively slowly compared to electrons of similar energies. Also, the fact that they have twice the charge of an electron makes them even more eager to interact with their
surroundings. Your outer layer of dead skin is enough to stop most \( \alpha \) particles, as would a piece of paper. When people talk about the dangers of plutonium (usually \(^{239}\text{Pu}\)), they don’t mean that standing near a block of it would be deadly. The problem is that inhaled particles of plutonium can lodge in the lungs, and that tissue is more sensitive (and alive!). A short range also means that the several MeV of energy will be deposited in a very small region, meaning that damage is more likely. Although this is very undesirable for imaging purposes, it is ideal for the treatment of some cancers (known as radiotherapy), since it is a localized form of destruction.

There’s another idea from quantum physics that proves helpful in understanding the atom, the nucleus, and \( \alpha \) decay, and that is the uncertainty principle.

**Heisenberg’s Uncertainty Principle**

The uncertainty principle can be stated as: we can’t measure both the position of a particle and its momentum to arbitrary accuracy at the same time. There is nothing like this in Newtonian physics – all of the uncertainty of measurement there comes about because of imperfections in our equipment. The uncertainty principle goes beyond saying that we don’t have perfect equipment, it says instead that we can’t have perfect equipment. To examine this more closely, look at what happens if we want to measure the position of an electron. How do we do that? At the most fundamental level, we look at it, which means we have to bounce photons off of it. In doing that, though, we transfer some momentum to the electron. If we want to measure the electron’s position to within one nanometer, for example, we need to use light with a wavelength of less than one nanometer. What momentum will the light have? We can use our previous results to see that

\[
\text{if } \lambda < 10^{-9} m, \quad p = \frac{h}{\lambda} \quad \text{so} \quad p > \frac{h}{10^{-9} m}
\]

Getting a better measurement of position means using light with a smaller wavelength, but that means each photon has a high momentum. Some (unknown) fraction of that momentum will be transferred to the electron, so measuring positional information causes us to alter, and therefore lose, momentum information. This is not obvious because \( h \) is so incredibly small. For macroscopic things (and even for many microscopic things), the momentum transferred by a visible-light photon is a ridiculously small fraction of the object’s total momentum. Also, we don’t often try to measure things to within a wavelength of visible light. The limits imposed by the uncertainty principle are not likely to affect the design of rulers anytime soon! We can write the formal expression for the uncertainties in momentum and position (written with deltas) as

\[
\Delta p \Delta x \geq \frac{h}{4\pi}
\]
This tells us that the uncertainty is at least Planck’s constant divided by 4 π. No measurement can do better than this. We have a tradeoff between position measurements and momentum measurements – we can in principle find the position of a particle down to any accuracy, but the photon we hit it with will give it a correspondingly large increase in momentum.

There is also an uncertainty relationship between energy and time. It basically says that we are allowed to violate conservation of energy (!) very briefly and “borrow” some energy from the vacuum, as long as we return it very quickly. The product of the energy uncertainty and the time uncertainty satisfies the same inequality as above:

$$\Delta E \Delta t \geq \frac{\hbar}{4\pi}$$

What this says is that the vacuum is to energy like a bank is to money. If we want to grab a large amount of money from the bank, we have to put it back very quickly if we’re going to do it before it is discovered. On the other hand, if we want to get a penny out of the bank, we can probably keep it for quite some time before it is noticed.

It’s strange and probably a little disturbing to see that one of the things so important in physics (conservation of energy) seems to be disappearing. Before you get too worried, do a sample calculation: if you only want to borrow some energy for a picosecond (10^{-12} seconds), how much can you have? Plugging 10^{-12} seconds in for \Delta t in the formula above gives 5.3 x 10^{-23} J, or about 3.3 x 10^{-4} eV. Not exactly a huge amount of energy!

One of the consequences of the uncertainty principle is that certain features of classical mechanics are radically altered. For example, if we imagine a ball bouncing inside a drinking glass, it is very straightforward to find out whether it will escape or not. If the kinetic + potential energy of the ball is less than the potential energy the ball would have at the rim of the glass, it absolutely won’t get out. The uncertainty principle allows the ball to “borrow” a small amount of energy from the empty space around it and return that energy once the ball has gone over the side. Since the time that energy can be borrowed is inversely proportional to the amount borrowed, and Planck’s constant is so small, we won’t see this happen with a ball & drinking glass.

The probability of penetrating a barrier drops very quickly as the barrier’s thickness increases. Also, if the barrier is very large compared to the particle’s energy, the probability of penetration drops rapidly. In the smoke detector example, the process can be modeled by assuming the α particle is trapped inside the nucleus like a particle in a well. The α particle bounces back and forth within the nucleus until (by random chance) it grabs a small amount of energy from the vacuum, climbs the side of the well, and appears outside the confines of the nucleus. Once it’s on the outside, the repulsive force between the two protons of the α particle and 93 remaining ones in the nucleus cause the α particle to rocket away and be detected by circuitry in the smoke detector.
For this reason, the energy of the escaping α tends to increase as the Z of the nucleus increases.

Something interesting to notice is that the energy borrowed really is returned to the vacuum. We can calculate the energy an alpha particle needs to have to escape its confinement by the nucleus, and we find that the emitted alpha particle has a lower energy than this. If the borrowed energy didn’t have to be returned (and therefore conservation of energy could be violated on the large scale), the emitted alphas would be ejected much more quickly than they really are.

This principle also explains the reason that atoms are stable. The electron can’t radiate photons and fall into the nucleus because that would make its position more certain (since the nucleus is smaller, we’d know where it is to within $10^{-14}$ m or so instead of $10^{-10}$ m), which would mean its momentum would be more uncertain, making it potentially larger. Increased momentum means increased kinetic energy. A smaller value of position, on average, would give a more negative potential energy but a more positive kinetic energy. The atom’s size is the result of a balancing act between confining an electron to a tiny volume (which the Coulomb force would like to do) and keeping its momentum and kinetic energy from increasing wildly.

**β decay**

The remaining kind of nuclear decay for us to examine is β decay. The three types of β decay are known as **electron emission**, **positron emission**, and **electron capture**. Each of these processes will change Z by one unit as a proton or neutron becomes a neutron or proton. The value of A does not change during β decay. If the balance of neutrons and protons is displaced from its ideal value, β decay can bring it back to a more stable ratio.

In electron emission, a down quark in a neutron becomes an up quark, changing the neutron into a proton. This process is mediated by the weak nuclear force, the details of which are not important to us. We can describe this decay by

$$n \rightarrow p + e^- + ?$$

The question mark is there because every decay must satisfy various conservation laws. In addition to the conservation of charge (zero on the left = +1 + -1 on the right), we have the conservation of baryon number and the conservation of lepton number. Baryons are particles composed of 3 quarks, like the proton and neutron, and have baryon number = +1. If we get three antiquarks together, they will form an antibaryon (the antiproton and antineutron being examples) which will have a baryon number of −1. We can see that the mystery particle above can’t be a baryon and it must have a charge of zero. Leptons are fundamental particles just like the quarks are. The difference is that, since leptons are “colorless”, they don’t feel the strong force as quarks do. Just as we have 6 flavors of quark, there are 6 flavors of lepton. They are
the electron, muon, tau particle (rarely called a tauon) and three neutrinos, one associated with each of the other three leptons.

The quarks are grouped into families of two, in the manner below

\[
\begin{pmatrix}
  u \\
  d \\
  c \\
  s \\
  t \\
  b
\end{pmatrix}
\]

The upper quark in each family has a charge of \( +2 \, e/3 \) and the lower quark has a charge of \( -e/3 \). The leptons are similarly grouped as shown below

\[
\begin{pmatrix}
  e^- \\
  \nu_e \\
  \mu^- \\
  \nu_\mu \\
  \tau^- \\
  \nu_\tau
\end{pmatrix}
\]

Each of the upper leptons has a charge of \(-e\) and each of the lower leptons (the neutrinos) is electrically neutral. All of these have a lepton number of +1. Each family has larger masses than the family to its left. For this reason, we can think of the \( \mu \) and \( \tau \) particles as “heavy electrons”. The neutrinos, though, are very odd particles. They have only the tiniest chance of interacting with matter, and almost every neutrino produced by the Sun and aimed at the Earth will pass through the whole planet without interacting.

After many years of careful measurement, it has been determined that neutrinos have rest masses well under 1 eV (the electron, the smallest non-neutrino matter particle, has a mass of 511,000 eV). All quarks and leptons have antiparticles, and the neutrinos are no exception. Just as antibaryons have a baryon number of -1, antileptons have a lepton number of -1. Also, leptons have zero baryon number, just as baryons have zero lepton number.

With all that in mind, can we now predict what particle is represented by the “?” in the decay scheme above? We know it must be a neutral antilepton, which leaves us with the antineutrinos. (As it turns out, the lepton numbers for each family are conserved individually in this interaction, so it has to be an electron-antineutrino, as shown below).

\[
n \rightarrow p + e^- + \bar{\nu}_e
\]

The bar over the neutrino symbol indicates that it is an antiparticle. This is the preferred decay scheme for \(^{137}\text{Cs}\), for example (see if you can work out what it must decay into). This is also what happens to an isolated neutron. Oddly enough, the neutron is only stable when it is part of a nucleus. If left alone, half of a sample of isolated neutrons will decay in about 10 minutes. When this \( \beta \) decay happens in the body, the antineutrino
will escape with virtually no chance of interacting with the patient or anything else on Earth.

Nuclei can also decay via positron emission, also called $\beta^+\$ decay. This process is similar to the one above, but now we have

$$p \rightarrow n + e^+ + \nu_{e^-}$$

Notice the changes relative to the first decay. Since the proton has a positive charge on the left, something on the right must have a positive charge, which is why we get a positron rather than an electron. Also, since the positron has a lepton number of $-1$, this is balanced by an electron-neutrino rather than an antineutrino.

Unlike in the case of the neutron, this process cannot happen outside of a nucleus, since the total mass-energy on the right is larger than that on the left. This is only possible when the proton can “borrow” energy from somewhere else, like the nucleus to which it belongs. As far as anyone has been able to determine, protons are stable. If proton decay is even possible, as some unified theories predict, experiments show that the half-life of the proton must be larger than at least $10^{32}$ years!

The process of $\beta^+$ decay is very important in medical imaging, as it forms the basis for positron emission tomography, or PET scans. We’ll examine these in greater detail later. Examples of nuclei which undergo $\beta^+$ decay include $^{18}$F, $^{15}$O, $^{11}$C and $^{13}$N.

Finally, just as $\beta^+$ decay attempts to fix the problem of having too many protons for the number of neutrons in a nucleus, electron capture is another way of doing the same thing (also known as a competing process). After positron emission, the atom will have lost an energy of at least $2\times 511,000$ eV, or about 1 MeV (half of this goes into the positron that is created, and the other half represents the electron that the atom will lose). If a nucleus has too many protons but energy of the product nucleus (also called the daughter) that would be produced by positron emission is less than about 1 MeV below the parent nucleus, electron capture is the only way the proton ratio can be adjusted.

This is most common in heavy elements such as $^{83}$Rb. In these elements, the outer shell electrons tend to compact the inner electron orbitals so that the innermost electrons spend a lot of time inside the nucleus itself, making capture that much easier. This competing process is undesirable from an imaging standpoint, since the PET scanner is looking for positrons rather than evidence of electron capture (abbreviated as EC). After EC has occurred, there will of course be an inner shell vacancy, so an electron from a higher shell will quickly move to fill it, resulting in either the emission of characteristic X-rays or Auger electrons. In either case, the dose to the patient is present with no chance of imaging it.
The electrons and positrons produced in $\beta$ decay typically have very short ranges. They are charged and will lose energy quickly in interactions with nearby atomic electrons (and, to a lesser extent, nuclei as well). Although lead is a good shield in terms of its ability to stop electrons, we already know what is likely to happen to high-energy electrons that encounter high-Z materials like lead: the production of X-rays via the Bremsstrahlung process. This could be a bigger problem than the electrons themselves in some cases! Low-Z shielding like plastics are usually used to stop electrons, since they aren’t very penetrating anyway and the low value of Z makes X-ray production extremely unlikely. Incidentally, the poor penetrating ability of electrons is the reason why X-ray tubes must be evacuated. It’s also the reason that, like $\alpha$ particles, $\beta$ decay is useful in radiotherapy (poor penetrating power = lots of energy deposited in a small, predictable volume).

$\gamma$ Decay

One of the most important kinds of decay for medical imaging purposes is $\gamma$ decay. If two nuclei have the same number of neutrons and protons but different energies (i.e., one is in an excited state), they are called isomers and the excited state is usually represented by putting an “$m$” for metastable after the “A” superscript:

$$^{99m}Tc \rightarrow ^{99}Tc + \gamma$$

The neutrons and protons in a nucleus can be thought of as occupying energy levels or shells, just as the electrons around an atom do. The major difference is that, while electronic transitions may involve energies of a few eV, nuclear transitions typically involve energies of millions of eV. This is the fundamental reason why the “atomic” bomb and hydrogen bomb are in a completely different class from any ordinary chemical explosive.

If the nucleus ends up in an excited state (usually as a result of a previous $\alpha$ or $\beta$ decay), it will emit a $\gamma$ to get back to its ground state. The $\gamma$ can escape completely or it can interact with one of the atomic electrons on the way out. If it hits one of the electrons, it will certainly ionize it, and this process is called internal conversion. This is analogous to the production of an Auger electron when X-rays are involved. In each case, the photon can either leave without interacting or it can give all its energy to an electron, ionizing it in the process. For imaging purposes, giving the energy to an electron is not good. Our equipment is designed to catch photons, and any competing process will only add to the patient’s radiation dose without aiding image quality.

Gamma rays are, in general, the most penetrating of the three types of radiation we have seen. Depending on the energy of the photon, the $\gamma$ can penetrate large thicknesses (meters, in some cases) of concrete or steel. Energies this large would be undesirable in an imaging application; we have to have some way to stop the photons so we can detect them. The most commonly used isotopes for medical imaging produce gammas in the range from about 80 keV to around 600 keV.
Radioactivity

We have looked at a few different kinds of radioactive decay, but we also need to know something about the rate of decay. Will all of the radioactive nuclei in a sample decay at once, or at a linear rate (50 decays per second until they’re all gone, for example), or something else?

The decay of atoms is very different from the gradual aging and death of plants and animals. For radioactive atoms, we define a half-life (written as $T_{\frac{1}{2}}$) as the amount of time necessary for half of a given sample of atoms to decay. This is really a statistical measure of the chance for any given atom to decay in a certain period of time, and since it is statistical, we will only be able to apply it to large numbers of atoms (not usually a problem, since a billion billion atoms would only be micrograms of most elements). In the same way, if we flip a coin twice and get heads both times, that doesn’t prove it’s not a regular coin. If we do it many, many times and get a significant deviation from 50% heads, we may have a trick coin.

One of the big differences between atoms and living things is that the chance of decay is independent of the age of the atom. For example, $^{14}$C has a $T_{\frac{1}{2}}$ of about 5700 years. The independence of age means that a 20,000 year-old $^{14}$C atom has as much chance of decaying in a given time period as a brand new $^{14}$C atom. This also means that the number of atoms decaying at any point in time depends linearly on the number of atoms which have not yet decayed. If $N$ is the number of atoms at any given time, the rate of decay (measured in decays per second and known as the activity of the sample) is given by

$$\frac{\Delta N}{\Delta t} = -\lambda N$$

where $\lambda$ is a constant which depends on the particular type of atom, and is called the decay constant. Notice the negative sign – that’s what tells us the sample is getting smaller over time. In the SI system, we measure activity in becquerels (Bq) where 1 Bq = 1 decay per second (a very low activity). An older unit is the Curie, which is equal to 37 billion decays per second.

Anyway, to find the amount of an element remaining after time $t$, we need to solve the equation above. When we do that, we get that

$$N(t) = N_0 e^{-\lambda t}$$

where $N_0$ is the amount of the element you had when you started your watch ($t = 0$). Since we know that when $t = T_{\frac{1}{2}}$, $N$ will equal $N_0 / 2$, we can use that to relate the decay constant to the half life and we get
\[
T_{1/2} = \frac{\ln 2}{\lambda} \approx \frac{0.693}{\lambda}
\]

Scientists use this idea to find the dates when a given sample of material was formed. This process is used to find the approximate year of death of animals and plants, which take in $^{14}$C along with ordinary $^{12}$C while they are alive. Once dead, the amount of $^{14}$C starts to decay. If measurements show that only 3.5% of the original amount of $^{14}$C in an organism remains today, we can solve the math above to find the time of death:

\[
N(t) = N_0 e^{-\lambda t}
\]

\[
0.035 N_0 = N_0 e^{-\lambda t}
\]

\[
\ln(0.035) = -\lambda t
\]

\[
t = \frac{-\ln(0.035)}{3.83 \times 10^{-12} \text{ s}^{-1}} = 8.75 \times 10^{11} \text{ s} \approx 27,750 \text{ years}
\]

Sometimes, a radioactive element decays into a daughter atom that is itself still radioactive. In this case, we have a radioactive decay series. The decay of $^{238}$U continues until it becomes $^{206}$Pb, meaning there are several $\alpha$ decays (along with other kinds of decay). This kind of decay is responsible for Earth’s supply of helium, which does not combine chemically with anything and has such a high average velocity at normal temperatures that it quickly escapes Earth’s gravity when released from its underground deposits.

The half-life of an isotope is one of many things that must be taken into account in the process of isotope selection. We want to choose something that will decay with reasonable speed so that we can complete the imaging protocol in about an hour or less (this is for two practical reasons: first, it would be prohibitively expensive for a nuclear medicine clinic to be occupied all day imaging a single patient, and secondly, no patient will be willing and/or able to lie still for hours at a time). With those restrictions in mind, a long half-life would mean that most of the decays would happen after the imaging process is over and the patient has gone home. These photons obviously don’t contribute to the imaging process but just as obviously do contribute to the patient’s total radiation dose.

It seems that we might like something with a very short half-life. The problem then becomes the production and transport of the isotope before it decays completely. When we look in more detail at PET imaging, we’ll see that $^{15}$O (for example) is useful in some studies, but imagine if the production and imaging facilities were even 20 minutes apart:
that would be 10 half-lives, meaning that only \(1/2^{10}\) or about \(1/1000\)th of the isotope you produced would survive the trip to the patient. Clearly we need to have isotopes like this manufactured on-site, which will require the purchase of expensive equipment.

If we want to avoid the large expense, we are looking for an isotope with a half-life that is neither too long nor too short. We would like it to be “clean”, meaning we want \(\gamma\) emission only, since the emission of alphas and betas only adds to the radiation dose. Also, for reasons that will become more clear later, we would like a monochromatic \(\gamma\) spectrum (only one kind of \(\gamma\) emitted, as opposed to a range of several that may be emitted). This may seem like an impossible number of restrictions to satisfy, but it turns out that \(^{99m}\)Tc is almost ideal. It has a \(T_{1/2}\) of 6 hours and its spectrum is dominated by a single 140 keV \(\gamma\) emission. Notice that 140 keV is in the diagnostic X-ray range, which tells us that it’s an energy well-suited for imaging human tissue.

\(^{99m}\)Tc is itself produced from the decay of \(^{99}\)Mo (what kind of decay must this be?), which has a \(T_{1/2}\) of about 67 hours. This provides another stroke of luck for small hospitals, since a generator containing \(^{99}\)Mo is easily transportable and will last about a month, producing \(^{99m}\)Tc constantly. Because the parent half-life is much larger than the daughter half-life, the radioactivity of \(^{99m}\)Tc will grow rapidly as \(^{99m}\)Tc accumulates before leveling off and declining at approximately the same rate as the radioactivity of the parent \(^{99}\)Mo itself. Equilibrium between the two activities is reached after approximately 4 daughter half-lives, which is (conveniently enough) about 24 hours. This is why the generator in a nuclear medicine clinic is usually “milked” every morning for the day’s needs. The separation process is not difficult since Mo and Tc are different chemically. The problem of separating different isotopes of an element is the reason why it’s easy to design a low-yield nuclear weapon, but much harder to actually construct it.

**Isotope Production**

If we want to produce a specific radioisotope, we will typically start with a stable isotope and add either energy or nucleons to it in such a way as to make it unstable. One of the most straightforward ways to do this involves firing neutrons at a target. Since neutrons are about 2000 times more massive than electrons, they have little difficulty moving through the atomic cloud into the nucleus. Since they are uncharged, they are not repelled by the protons in the nucleus, and add themselves to it readily. The possible effects on the nucleus are varied and depend on the details of the collision (neutron energy and target composition). In addition to the three kinds of decay we’ve already studied, we could have proton emission, neutron emission, or a disruption of the nucleus into two or more large pieces (fission).

We can write the details of a particular reaction as

\[
\text{parent}(\text{bullet, shrapnel})\text{daughter}
\]

where parent is the target being bombarded by bullet, producing shrapnel and daughter. For example, we can write
if we mean that firing neutrons at a $^{10}$B nucleus gives us $\alpha$ particles and $^7$Li. Where do we get a neutron beam? Nuclear reactors are one source, since the fission occurring in the fuel rods produces excess neutrons.

Another common mechanism involves **charged particle activation**, where protons or larger nuclei are accelerated to high speeds by electric fields and slammed into a target. The charged particles can be accelerated by a **linear accelerator** or, more commonly, a **cyclotron**. The cyclotron is a circular accelerator that allows high energies to be reached in a small region (much more convenient than a kilometer-long linear accelerator).

The cyclotron relies on the fact that a charged particle moving in a magnetic field feels a force perpendicular to both the field and its velocity. This means that it will tend to move in a circle, and that the magnetic field will neither boost its velocity nor reduce it. A cyclotron is typically composed of two semicircular tracks (called “dees”, since they are shaped like capital Ds) with a small gap between them. An electric field between the dees serves to accelerate the particles each time the gap is crossed. Of course, this means that the direction of the electric field will have to change quickly so that the particles continue to be accelerated, as shown below.

The diagram above shows the arrangement for positively charged ions. A cyclotron can be either a positive ion machine or a negative ion machine. For a negative ion cyclotron, it is necessary have to have something like H⁻ (hydrogen with an extra electron) to accelerate, and the H⁻ ion is passed through a carbon foil to remove both electrons and create a proton beam before striking the target. The negative ion machine has a few benefits over the positive ion variety, including the ease of using more than one target at a time and the reduction in machine contamination with use.

The most commonly used isotope in PET imaging is $^{18}$F, which can be produced by (among other reactions) $^{20}$Ne(d,α)$^{18}$F where the d bullet is a deuteron ($^2$H) or $^{18}$O(p,n)$^{18}$F. The cyclotron is especially useful for PET imaging since we are trying to create
isotopes that will $\beta^+$ decay, which requires them to have an excess of protons. Because $^{18}$F has a half-life of about 2 hours, it can be made offsite and delivered to the PET scanner location, but it is reasonably common for a facility to have both a cyclotron and a PET scanner in the same place. This also makes imaging with things like $^{11}$C, $^{13}$N, and $^{15}$O feasible ($T_{1/2} \sim$ 20 minutes, 10 minutes, and 2 minutes, respectively).

The formula relating isotope production to the relevant variables involved is

$$ R_t = \sigma \ I \ n \ V \left( 1 - e^{-\lambda t} \right) $$

where $t$ is the irradiation time, $V$ is the target volume, $n$ is the number of target nuclei per unit volume, $\lambda$ is the decay constant of the product, $I$ is the beam intensity, and $\sigma$ is the cross-section for the reaction to occur. This tells us that the irradiation time should not be longer than $T_{1/2}$ for the isotope you're producing.

**Planar Nuclear Medicine**

The simplest kind of scan involving the radioisotopes above would be a standard planar nuclear medicine scan, such as a bone scan. In a bone scan, the patient will be injected with methylene diphosphonate (also known as medronate or MDP) which has been labeled with $^{99m}$Tc. After allowing a few hours for the distribution of the substance, the patient will be placed on a bed and imaged with a gamma camera. As its name implies, this is a device which is sensitive to gamma rays and will produce an image similar to the one below (from http://www.med.harvard.edu/JPNM/TF99_00/April18/BoneScan.gif)

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* p.32, *Nuclear Medicine Physics – the basics (5th edition)*, Ramesh Chandra, Williams & Wilkins
“Ant” stands for anterior, or from the front, and “Post” stands for posterior, or from the rear. Although this looks quite similar to an X-ray, it’s produced by gamma emission and it clearly shows a difference in the patient’s knees. The increased uptake in the right knee in this case indicated a cancerous tumor. An X-ray, showing structure instead of function, might have been unable to differentiate between the tumor and the “normal” tissue.

### The Gamma Camera

The gamma camera is a device which is sensitive enough to detect individual gamma rays. It is composed of a crystal of NaI doped with TI (meaning small amounts of thallium have been added to the NaI crystal). The crystal is known as a scintillator, meaning it produces visible light after the recombination of ions produced by the passage of ionizing radiation through the crystal. NaI(Tl) has a high intrinsic efficiency, which is a measure of the number of detected rays divided by the number of incident rays. It also has a short dead time, which is the time it takes the detector to
complete the detection event. This is in the 5 µs range for NaI(Tl), which means it can work with a reasonably high count rate before problems begin to occur.

The problem with higher count rates is that the system can only respond to so many events per second; at low count rates, we should see a linear relationship between the number of interactions with the crystal and the number reported by the system. The dead time means that this proportionality can’t go on forever – at some point, the interactions will be happening so quickly that the number missed will increase to unacceptable levels. You can think of the dead time as being like the time necessary for a mousetrap to spring and be reset; any mouse that comes by after an earlier mouse has triggered it but before it has been reset will be missed.

Detectors can also be classed as paralyzable and nonparalyzable. This describes the reaction of the system to events happening during the dead time period of a previous event. If a detector (we’ll assume it has a dead time of 100 µs) is nonparalyzable, that means that the 100 µs clock starts as soon as radiation interacts with the crystal. If another gamma comes along 32 µs after the first, it will not be counted. A third gamma arriving 101 µs after the first would be recorded, though. For a paralyzable detector, the second gamma ray would still be lost, but it would reset the 100 µs clock, meaning the third gamma would also not be counted (and would also reset the clock).

This means that the detection rate of a nonparalyzable detector has a maximum count rate and adding more radiation will not increase that count rate. For the paralyzable detector, adding more radiation will actually decrease the count rate. Either of these cases is bad, since it means we don’t get an accurate count of the decays. Along with radiation exposure, this is one reason why we don’t try to speed up the imaging process by giving more of the radioisotope to the patient.

When a gamma ray hits the NaI crystal, it will produce many visible photons. These photons travel through the crystal until they reach a bank of photomultiplier tubes (PMTs). PMTs are sort of like solar cells on steroids. When a photon hits the face of the PMT, it has about a 10% chance of causing an electron to be ejected from the metal (the photoelectric effect again). Inside the PMT, the electron is subject to an electric field that accelerates it from one stage (called a dynode) to another. When an electron hits a dynode, it releases a larger number of electrons into another electric field, which pulls those electrons into another dynode, which again magnifies the number of electrons involved. A single photon liberating a single electron quickly produces an electrical pulse of measurable current. The PMT typically uses several stages of dynodes to amplify a pulse by a factor of around one million. The back of the NaI crystal is covered as completely as possible by these PMTs to increase the chance that a scintillation anywhere in the crystal will not be missed.

Because PMTs are usually cylindrical, this means that their round faces can’t completely cover all parts of the crystal, but if packed in a hexagonal manner, they can minimize the problem, as shown below.
The precise location of a scintillation can be found by looking at the signal strength from each PMT. The interaction site should be nearest to the PMT with the largest signal, and the signals of its neighbors allow more precise position information to be calculated (this idea is known as Anger logic, after the camera’s inventor). The ability of the camera to accurately locate the site of the interaction in this manner is known as its intrinsic resolution. For the 140 keV γ rays of ⁹⁹ᵐTc, this resolution is around 4 mm. This is one of the reasons a high-resolution X-ray or CT scan (with sub-millimeter resolution) looks so good compared to the typical nuclear medicine scan.

Because we would like to exclude scattered γ rays from our image, and because scattering reduces the energy of the γ ray, it would be nice if we could measure the energy of each γ and tell the circuitry to ignore those with energies less than 140 keV. It turns out that we can do this, to a limited extent. The source of the limitation is the energy resolution of the camera. If we had a way to magically discriminate between a 140 keV γ and a 138 keV γ, our pictures would look much better.

Unfortunately for us, the energy resolution of the gamma camera is such that we generally need to open an energy “window” which is about 20% the size of the γ ray itself, meaning we allow energies between ~125 keV and 155 keV to be counted as 140 keV γ rays. What is the source of this poor resolving ability? It all boils down to the (relatively) small number of visible photons produced by a single γ. To see how small that number is, we first take the 140 keV of the γ and divide it by 3 eV (about the energy of a single visible light photon) and get about 46,000 photons. Of course, only about 15% of the energy from the γ becomes visible light, and about 50% of that light will be going the “wrong” way (away from the PMTs). Finally, only 10% of the visible photons that make it to the PMT are successful in liberating an electron. Taken together, that means that a single γ will probably result in the production of about 350 (46,000 * 0.15 * 0.50 * 0.10) electrons. A random process like this obeys what are known as Poisson statistics (see sidenote below). In this kind of process the standard deviation of the data will be approximately equal to the square root of the mean (√350 or about 18 in this case). The energy resolution is then found from
\[ \frac{\delta E}{E} = \frac{2\sqrt{2 \ln 2 \sqrt{350}}}{350} \approx 12.5\% \]

We would need a better scintillator to improve this resolution, and NaI is pretty hard to improve on in other respects.

**Poisson Statistics and the Binomial Distribution** *(not vitally important to know)*

If you flip a fair coin, you expect to have a 50% chance of getting heads and a 50% chance of getting tails. What if you flip it twice? If you don’t get one head and one tail, you probably won’t be amazed. In fact, it’s pretty easy in this case to list all of the possible outcomes (HH, TT, HT, TH) and notice that you get “proof” the coin is fair in only half of the cases. Of course, we know it would take more than 2 heads or 2 tails in a row to amaze most people.

What about if you flip the coin 100 times? Getting 100 heads in a row from a fair coin would definitely be amazing, but how likely are you to get exactly 50 heads and 50 tails? You might guess instinctively that a perfect 50-50 split is also not very likely. The probabilities in this case can be calculated using the **binomial distribution**. For a fair coin, we could write the number of ways we could possibly get \( k \) heads in \( n \) flips as

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

where the exclamation point (factorial symbol) means we should multiply all the whole numbers between 1 and the preceding number: \( 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120 \). Checking this for our two-flip example, we see that it does predict that we could get a single occurrence of heads in two ways (first flip or last). If we wanted to turn this into a percentage, we could divide the formula above by the possible number of different outcomes (where order is important, so HT is not the same result as TH). That number would just be \( 2^n \).

Using a computer, we can find that the number of ways we can have the coin land on heads 50 times in 100 coin flips is 100,891,344,545,564,193,334,812,497,256. The total number of ways 100 flips could happen is \( 2^{100} \), or 1,267,650,600,228,229,401,496,703,205,376. Dividing those two gives us about an 8% chance that we’ll get exactly 50 heads and 50 tails. That means a 92% chance we’ll get something else. Of course, 49 heads and 51 tails is a lot more likely than 0 heads and 100 tails (only one way that can happen). In fact, we can look at a plot of the relative likelihood of a given outcome for a given number of flips.
Notice how the distribution gets progressively taller and narrower, meaning that the chance of getting a result far away from the most likely result starts to drop dramatically. For a more general example, you might think the chance of getting 4 heads in 10 flips is the same as getting 40 heads in 100 flips or 400 heads in 1000 flips. The real numbers are below
The Poisson distribution is an easier to calculate generalization of this to large numbers of flips and coins that are inherently not fair (75% chance of heads on a given throw, for example).

<table>
<thead>
<tr>
<th>Flips</th>
<th>Heads</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>20.5%</td>
</tr>
<tr>
<td>100</td>
<td>40</td>
<td>1.08%</td>
</tr>
<tr>
<td>1000</td>
<td>400</td>
<td>0.0000000046%</td>
</tr>
</tbody>
</table>

**Collimators**

The remaining part of the equipment for a nuclear medicine scan that we haven’t discussed is the **collimator**. This is a piece of high-Z material (typically lead or tungsten) that sits on top of the crystal in between the patient and the camera. The collimator’s purpose is to reject photons that are not traveling in a certain direction. The reason this is necessary is that we measure the interaction location in the crystal, but what we really want to know is the decay location in the patient. The collimator provides us with a way to increase the likelihood that we will accurately locate the true position of the decay event.

The simplest type of collimator looks very much like a section of honeycomb made from lead rather than wax. The holes are shaped like hexagons and each hole is a tube parallel to every other hole. This is called (not surprisingly) a **parallel-hole collimator**.

![Collimator Diagram](image-url)
Below is the view of a parallel-hole collimator as seen from the patient’s perspective. For a low-energy general-purpose (LEGP) collimator, typical dimensions would be a flat side to flat side diameter of about 1.4 mm, hole length of around 25 mm, and septal thickness (thickness of the gray wall of lead separating two hexagons) of about 0.18 mm.

The length and hole diameter together determine how selective the collimator will be. For a long, narrow hole, only $\gamma$ rays in a very small range of angles will be admitted to the crystal. Short, wide holes will allow a wider range of $\gamma$ rays to be detected (see below).

There are tradeoffs to be made here. An overwhelming number of photons (>99.9%) generated in the patient will never make it to the detector. Some of this is to be expected since the radiation should be emitted in all directions, and the solid angle occupied by the gamma camera is small (the solid angle could be calculated by dividing the area of the camera’s face by $4\pi (\text{distance to camera})^2$).
Many of the photons that are aimed at the camera are stopped by the lead in the collimator. We will be much more sure of the direction of the decay location when we have a high-resolution collimator (meaning long, thin holes) on the camera, but we'll get very few of these photons to make our image. In general, we can say that the efficiency of a collimator (a measure of how many photons it will allow through) is proportional to the square of the resolution (for resolution, a larger number means a picture with lower detail).

Another consideration in collimator selection and design is the thickness of the septa. If the septa are too thin, they will be unable to serve their purpose by stopping misdirected $\gamma$ rays. If, on the other hand, the septa are too thick, we will find 1) reduced efficiency since the holes are effectively smaller and 2) if the septal thickness approaches the intrinsic resolution of the camera, we'll start to get images where the hexagonal pattern of the collimator is visible, which we certainly don't want. This is why energies are specified in the names of the collimators, since a collimator designed to stop 140 keV photons can have much thinner septa than the ones necessary to image with high-energy isotopes such as $^{137}$Cs which produces a 662 keV gamma ray. The septal thickness is chosen to minimize penetration, but not to completely eliminate it because of the restrictions above.

**Nuclear Medicine vs. X-rays**

We've already seen some of the main differences between these two imaging modalities – structure vs. function and transmission vs. emission. We also notice that X-rays seem to be much more clear than nuclear medicine scans. The resolution differences have already been discussed, but another reason is the nature of the radiation dose. In an X-ray scan, all of the radiation is emitted while the detector (or film) is watching. In the typical NM scan using $^{99m}$Tc, we have the physical half-life of 6 hours coupled with the biological half-life of which describes the rate at which the radioisotope is diffused throughout the body or excreted. The effective half-life, which describes the combined effects of the physical and biological removal of the radioisotope, is given by

$$\frac{1}{T_{eff}^{1/2}} = \frac{1}{T_{bio}^{1/2}} + \frac{1}{T_{phys}^{1/2}}$$

In other words, while a glass vial containing $^{99m}$Tc would see the concentration of it decrease by a factor of about 1000 after 60 hours (10 physical half-lives), less than $1/1000^{th}$ of the $^{99m}$Tc administered to a patient will still be present in his or her body after 60 hours.

This makes things a little better for the NM process, but we still have a radiation dose delivered over hours to days when we are only realistically able to monitor it with the gamma camera for 30 minutes to an hour. We will get to see far fewer of the photons than we would in the X-ray case, but we are still limited by the total overall dose to the patient.
Additionally, being able to collimate the X-ray beam before and after it leaves the patient means we only send in X-rays that have a good chance of making it through. Since the radioisotope in an NM scan will emit radiation isotropically (in all directions equally), the detector will only be in a position to catch a small number of them. As we’ve seen, that small number is further reduced by the presence of the collimator. It’s a wonder we can get anything at all from a nuclear medicine scan.

The other major problem is the fact that an X-ray scan measures one thing: the attenuation of an X-ray beam of known incoming intensity. If we have a region of the film or detector which doesn’t receive many X-rays, we know that the combined μx for that path through the body is large. When we discuss CT scans, we’ll see how the μx term can be broken down into a high-resolution attenuation map of the body. This is the way we can decide if a large value of μx is due to a very high μ and a small x or a small μ and a large x, or whatever combination may be present.

Nuclear medicine scans are meant to measure another variable, which is the concentration of the radioisotope. Unfortunately, just because we’re not interested in attenuation doesn’t mean it disappears. We now have both problems – all the difficulties of attenuation in an X-ray coupled with the uncertain distribution of the radioisotope. Now, if we see a part of the detector that isn’t receiving much radiation, is it because μx is large or because the uptake of the radioisotope was small? This can turn what is essentially a solvable problem (the CT scan) into a horrible mess (the NM equivalent, called SPECT).

In recent years, this problem has been overcome to a large extent by combining both modalities. A patient getting a SPECT scan will commonly also get a CT scan. The CT scan gives us a map of the attenuation coefficient everywhere in the patient, which we can combine with the data from the gamma camera to find the true radioisotope distribution. Machines that can do both CT and SPECT or PET are called hybrids.