
**Units of Measurement**

One of the hardest things for new physics students to get used to is the fact that most physics professors seem to be obsessed with units of measurement. The reason for this is simple - if you don’t mention the units you’re using, your statements will usually have no content. For example, if a friend is traveling around the world and tells you that his hotel room was “4,000”, that means nothing. If it was 4,000 US dollars, it must have been a nice room. If it’s 4,000 lira or 4,000 yen, it might have been a phone booth.

There are no physically important quantities which can’t be measured somehow (except for things like \( \pi \) and other constants from mathematics rather than physics itself), and if they can be measured, they have units of measure. The basic metric units are meters (length), kilograms (mass), and seconds (time). Various prefixes attached to these units provide a shorthand way to create new units which are more appropriate to a given situation. When we discuss the size of light waves, we’ll use nanometers (nano = one billionth). If we’re talking about the distance from the Earth to the Sun, gigameters (giga = one billion) would be more appropriate. Here are some of the most common prefixes in the metric system, and the values they correspond to:

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tera</td>
<td>( 10^{12} )</td>
</tr>
<tr>
<td>Giga</td>
<td>( 1,000,000,000 = 10^9 )</td>
</tr>
<tr>
<td>Mega</td>
<td>( 1,000,000 = 10^6 )</td>
</tr>
<tr>
<td>Kilo</td>
<td>( 1,000 = 10^3 )</td>
</tr>
<tr>
<td>Centi</td>
<td>( 0.01 = 10^{-2} )</td>
</tr>
<tr>
<td>Milli</td>
<td>( 0.001 = 10^{-3} )</td>
</tr>
<tr>
<td>Micro</td>
<td>( 0.000001 = 10^{-6} )</td>
</tr>
<tr>
<td>Nano</td>
<td>( 0.00000001 = 10^{-9} )</td>
</tr>
<tr>
<td>Pico</td>
<td>( 10^{-12} )</td>
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These prefixes are among the very few constants you’ll be expected to memorize.

**Dimensional Analysis**

Units of measurement are also involved in dimensional analysis. Dimensional analysis is basically a way to look at your answer and see if it makes sense, and it can sometimes tell you exactly what piece you’ve left out. For example, if we’re talking about a velocity, we know that it must be expressed as \([\text{Length}] / [\text{Time}]\). We’re used to seeing things like miles per hour for this, but the US is essentially alone in sticking to the English system of units. In the metric system, velocity would be meters / second. Areas would be measured in square meters (area of a square = side length\(^2\), area of a circle = \( \pi r^2 \), etc.). If you’ve ever bought carpet, you probably bought it in square yards instead. Volumes are measured in cubic meters (very large – dumpsters may be measured this way) or liters (much smaller – a cube with a side length of 10 cm).
As an example of how this can help you, you may do a lab experiment which involves hanging weights from a string and then shaking it back & forth to see how quickly waves travel down it. You’ll see that the velocity (m/s) is directly related to the square root of the tension in the string. We’ll worry about what tension is later, but for now, all you need to know is that the units of tension can be written as kilograms * meters / seconds$^2$. Just by looking at this, you know something must be missing, as you can see below:

$$\frac{m}{s} = \sqrt{\frac{kg \cdot m}{s^2}} \rightarrow \sqrt{\frac{kg \cdot m}{s^2}} \neq \frac{m}{s}$$

We can look at this and see that we have a factor of $\sqrt{kg}$ that we don’t need, and we still need one more factor of $\sqrt{m}$ to get meters per second. What we’ll see later in the semester is that the missing piece is the mass per unit length of the string (i.e., is this dental floss, or heavy rope?). Can you guess where that would have to be in the formula to make the units work out?

**Unit Conversions**

Frequently, we need to go back and forth between different units when doing a problem. There are a few things to keep in mind. First, remember that we can only convert between units measuring the same thing – there’s no way to convert from meters to seconds, for example. Also, remember that the only thing we can ever do to an isolated quantity (220 miles to Atlanta, for example) is multiply by one. That sounds pointless, but in fact the whole idea of unit conversion lies in the number of ways you can write “one”. For example, 1 mile is equal to 1.609 kilometers. Below, we find the distance to Atlanta in km:

$$220\text{ miles} \times \frac{1.609\text{ km}}{1\text{ mile}} = 354\text{ km}$$

Notice how we wrote “one” as 1.609 km over 1 mile. That quantity is, in fact, equal to one just like 12 inches is equal to one foot. How did we know not to write it as 1 mile over 1.609 km? That would still be one, right? Look below at the difference.

$$220\text{ miles} \times \frac{1.609\text{ km}}{1\text{ mile}} = 354\text{ km} \quad \text{but} \quad 220\text{ miles} \times \frac{1\text{ mile}}{1.609\text{ km}} = 137\frac{\text{ miles}}{\text{km}}$$

Written the first way, the “miles”, which we want to get rid of, go away and leave only km behind. The second way, we get miles squared, as well as km on the bottom. That’s why we always write our “one” in such a way as to get rid of what we want gone, and replace it with what we want.

**Significant Figures**

It’s important to keep an eye on the number of figures to record after a calculation. If you’re trying to decide how to divide a kilogram of sand into 3 piles, you can use your calculator to find
out how much to put in each pile (hopefully, you don’t need to, but we’ll do it anyway). If you put $1 \div 3$ into your calculator, you’ll get back an answer of 0.333333 (depending on how many digits your calculator reports). If you look at this for a minute, you can see that the output has many more digits than the input. If you measured the sand carefully, maybe you know that it was 1,000 grams. Maybe you went to a chemistry lab to borrow a nice scale, and you’ve found that it’s 1,000,000 grams. Writing it this way indicates that you’re sure of the mass down to the last figure shown (the final 0 in “,000”). This says you know the answer to within one milligram (because if you measured your sand and got 1,000.001 grams, that would be one milligram of difference). If you only know the mass of the sand to within a milligram, you can’t then write down that each pile will contain $1,000.000 \times 0.3333333 = 333.33333$ grams, because the final “3” here would represent 30 micrograms, and your scale is not that accurate. In other words, the number of significant digits in your final answer is limited by the number of significant digits in your input. If you know the total mass to within a milligram, you’ll only know the mass of each pile to within a milligram (at best).

You may have noticed referees in football ignoring this fact: for the first three downs, the referees make an estimate with their eyes about where the ball should be placed. On the fourth down, the chains come out and the position of the ball is measured to within a fraction of an inch. It makes no sense to attach a very precise measurement to several other rough estimates. As another example, police frequently make damage estimates to the cars involved in a wreck. If the estimate for a car’s repairs is $4,000, does it really matter if the officer didn’t notice that your air freshener was damaged in the crash? Should he change his estimate to $4,000.98?

There are some relatively simple rules for deciding about the number of digits to keep during a calculation. If you are multiplying or dividing, the answer can only have the same number of significant figures as the least-precise number involved. For example, if you’re finding the area of this room, and you measure it to be 12.1 meters in one direction, and 15.255 meters in the other direction, your calculator will claim that the area is 184.5855 m$^2$. However, you can only keep 3 digits, because your first measurement only had that many. The fourth digit tells you how you’ll record the other three. In this case, the 4th digit is a 5, which means we’ll round the 3rd digit up. Your answer should be 185 m$^2$. Remember that this happens because of your uncertainty in the 12.1 m measurement. It could really be 12.05 m, or 12.1499… m, or anything in between. The thing to remember is that the calculator is not smart – it has no ability to evaluate and make conclusions – it’s just there to speed number crunching.

When you’re talking about addition and subtraction, the number should have the same number of decimal places (not necessarily the same as number of figures) as the least-precise number. For example, let’s say I want to know how far the top of a 40-story building is from the center of the Earth. The book says it’s 6378 km from the center of the Earth to the surface. A 40-story building is approximately 120 m tall. If I try to add these together, I need to put them in the same units:

\[
\begin{align*}
6,378 \text{ km} \\
+ \quad 0.120 \text{ km}
\end{align*}
\]

6,378 has the fewest number of decimal places (zero), so we have to limit the answer to zero decimal places (whole number). That means we round 0.120 km to 0 km. In other words, if the only data we have is the back of the book, we don’t know the distance to the center of the Earth well enough to make the distinction between the ground floor & penthouse of this building. We
can put the numbers in the calculator, and it will happily produce 6378.12 for the answer, but the answer can’t be trusted.

Chapter 2: Kinematics
Distance and Speed vs. Displacement and Velocity

Most people feel comfortable with the ideas of distance and speed. Distance just talks about the separation between two things, and speed just describes how fast we can get from one thing to another. Both of these quantities are known as **scalars**. Scalars are objects without direction: your car’s odometer measures distance (miles since it came from the factory) and its speedometer measures speed (how fast are you currently going). **Displacement** includes directional information. For example, if you were to drive your car back to the factory, its total displacement since it was built would be 0 km. You might have gone through 3 sets of tires and an engine rebuild, but its displacement is zero. Quantities with both magnitude and direction are called **vectors**, and they’re drawn as arrows. The pointy part of the arrow indicates the direction the quantity is going. To find displacement, put the tail of the arrow at the initial point, and draw a line to the final point, ending with an arrowhead. In the example of the car returning to the factory, the line has zero length. Pay careful attention to the distinction between **units** and **direction**. 75˚ C is a scalar, because it has no direction attached. 35 m East (or up, or down, etc.) is a vector because a direction is attached. **Velocity** is also a vector. There’s a big difference between driving 100 km/hr East and 100 km/hour West (very noticeable if you happen to be on the East-going side of the interstate). In three dimensions, vectors have 3 parts, called **components**. We usually only think of two of these if we’re driving – how far East, and how far North (negative numbers would then put us to the West or South). If we’re looking at an airplane, its velocity vector has 3 obvious parts: East/West, North/South, and up/down. The airplane may be moving 200 km/hr to the North, 300 km/hr to the East, and 10 km/hr up. If you continue in physics, you’ll see stranger cousins of vectors called **tensors**, but we won’t be using them. We will soon look at how to deal with vectors in more detail, but first, we should think about what we really mean by velocity.

We can talk about **average velocity** or **instantaneous velocity**. Your speedometer (if it gave directional information) gives instantaneous velocity. Average velocity is what you might calculate after a long day of driving (if you’ve gone 600 km West in 6 hours, your average velocity is 100 km/hr West). Each of these quantities can be useful, but you should always be aware of which one you need and why. If a friend tells you she averaged 100 km/hr on her drive, that is a summary of information. Maybe she drove the full 600 km in 6 hours with no stops and the cruise control set, or maybe she spent 10 minutes stopped for food and later had a high-speed chase with the police. If you only know the average, you’ll miss out on all the interesting stuff like that. If, however, your friend’s **acceleration** was zero on her whole trip (except for her initial start and final stop), the average velocity by itself tells the whole story.

The formula for instantaneous velocity is just
\[
v = \frac{dx}{dt}
\]

The average velocity formula uses finite instead of infinitesimal displacements and times, and is usually written with a bar over the v, like this
In this formula, $dx$ is the change in position from beginning to end, also known as the displacement. $dt$ is the change in time from beginning to end. As the size of the time interval $dt$ gets smaller and smaller, the distance traveled also gets smaller, and the velocity gets closer to its instantaneous value.

**Acceleration**

When an object’s velocity changes in magnitude and/or direction, we say that it has undergone an *acceleration*. Acceleration has both magnitude and direction, and is therefore a vector. The definition of acceleration is the change in velocity per unit time. As with velocity, we can define both an average acceleration and an instantaneous acceleration.

If acceleration is defined as change in velocity divided by change in time, we can write that as

$$a = \frac{dv}{dt}.$$
Kinematic Equations

There are 4 basic equations we’ll need to talk about the way things change position and change velocity. As with any formula you learn in here, you have to understand when it applies. These four equations are valid **only when the acceleration is constant**. Using the wrong formula for a given situation will give you the wrong answer almost every time. The formulas we’ll need are:

\[ x = x_0 + vt \]
\[ v = v_0 + at \]
\[ x = x_0 + v_0 t + \frac{1}{2} at^2 \]

Each of the three equations above can be found by integrating the earlier differential expressions for \( v \) and \( a \). We can combine them to find the fourth kinematic equation:

\[ v^2 - v_0^2 = 2a(x - x_0) \]

The first of these four equations just describes how distance is covered as an object moves **with constant velocity**. This is a very important thing to remember – if an object is accelerating, this formula will not accurately tell you what’s going on. In this formula, the \( x \) is the final position of the object after it has moved with velocity \( v \) for time \( t \). The final position is therefore \( v*t \) plus whatever the initial position was (represented by \( x_0 \)). This is the formula you would use if you were on a road trip and fell asleep for an hour (not while driving, hopefully). You’d wake up, look at your watch, and realize you were probably 65 miles past wherever you fell asleep.

The second equation gives the final velocity \( v \) of an object after it has moved with a constant acceleration \( a \) for a time \( t \). As before, this velocity change must be added to the initial velocity \( v_0 \).

This is one way to find the speed of a falling object – you just need to know how long it fell, how fast it was going in the first place, and how quickly it accelerates when falling.

Notice the similarity between the first and third equations. In fact, we don’t really need the first equation as long as we have the third one, because the first one assumes constant velocity, or zero acceleration. If we put in a value of zero for \( a \) in equation 3, equation 1 pops right out. The advantage of equation 3 is that it is still valid even when \( a \) is not zero. This is the formula we use to find out how much distance is covered by an accelerating object. For example, if you’ve ever dropped a penny off of a tall bridge & timed it, you can use that time to find the bridge height. If it takes 4 seconds for the penny to hit the ground (and we neglect air resistance), we can fill in the numbers to find the height. First, we’ll choose to set your hand as the zero displacement point. That means \( x_0 \) will be equal to zero, and disappear. Since we’re not throwing it, but just dropping it, its initial velocity \( v_0 \) is also zero, so the \( v_0*t \) term also disappears. We know \( a = 1 \) \( g = 9.8 \text{ m/s}^2 \), so we’re set.

\[ x = x_0 + v_0 t + \frac{1}{2} at^2 = 0 + 0*(4 \text{ sec}) + \frac{1}{2}(9.8 \text{ m/s}^2)(4 \text{ sec})^2 = 78.4 \text{ m} \]
So, you’re over 250 feet in the air!
The fourth equation is useful when you want to relate initial and final velocities to acceleration and the distance over which that acceleration occurred. If we look at another car example, the fastest dragsters in the world can complete a run (usually ¼ of a mile) at a speed of around 300 mph! We can find the average acceleration with equation four. Keep in mind that we can only find the average because our data is limited to the initial and final velocities and the distance covered. The instantaneous acceleration is certainly changing throughout the whole trip down the track. This will tell us that the driver experiences an acceleration at least equal to the average, but probably greater at some points. The distances and velocities become, in metric units:

\[
\frac{1}{4} \text{ mile} \times \frac{1609 \text{ m}}{1 \text{ mile}} = 402.3 \text{ m} \quad \frac{300 \text{ miles}}{1 \text{ hour}} \times \frac{1609 \text{ m}}{1 \text{ mile}} \times \frac{1 \text{ hour}}{3600 \text{ sec}} = 134 \text{ m/s}
\]

Our equation is now (remembering that the car starts from a dead stop):

\[
v^2 - v_0^2 = 2a(x_f - x_0) = (134 \text{ m/s})^2 - (0 \text{ m/s})^2 = 2a(402.3 \text{ m} - 0 \text{ m})
\]

Solving this equation gives us a value of 22.3 m/s² for the acceleration, or about 2.3 times the acceleration due to gravity! This equation is also very useful when you want to know how high something can be shot or thrown straight up. You know that the final velocity (which in this case means the velocity at its maximum height) has to be zero at the top of its flight, and you know that \(a = g\). You can either measure the height to find the speed of the throw, or if the speed is known, you can accurately predict the maximum height. (Notice that the object’s speed when it returns to your hand will be exactly the same as the speed when it left! This ignores air resistance, but it’s why shooting guns into the air is generally not a smart idea).

One of the biggest parts of understanding the kinematic equations is learning when to use which one. The first, for example, is not valid when acceleration is occurring. With practice, you’ll learn how to apply these to solve various problems.

While we can’t use the kinematic equations in their present forms if \(a\) is not a constant, we can still find \(a\) or \(v\) working in the other direction. For example, if you are told that the formula for an object’s position as a function of time is

\[
x(t) = pt^4 + qt^2 + rt + s
\]

you can still find \(v\) by taking the derivative with respect to time. You’ll get

\[
v(t) = 4pt^3 + 2qt + r
\]

The acceleration involves one more time derivative, so we get
\[ a(t) = 12pt^2 + 2q \]

We now know that, at time \( t = 3 \) seconds, we should find that \( x = 81p + 9q + 3r + s \), \( v = 108p + 6q + r \), and \( a = 108p + 2q \). You could turn these into numbers if you were given the values of \( p, q, r, \) and \( s \). By the way, what are the units for these variables? Is there any way to figure them out?

Free Fall

The kinematic equations above can be used to analyze the behavior of objects in free fall (falling under the influence of gravity alone) near Earth. Basically, all we do is replace the \( a \) for acceleration with the known value of \( g \) (9.8 m/s\(^2\)), except that we make the 9.8 m/s\(^2\) negative, since the object falling is accelerating downwards, towards the ground. Even if you fired a bullet straight up, it would be accelerating downward the entire time. This is sometimes a hard concept for students to keep in mind – acceleration and velocity don’t have to be in the same direction. When we say all things accelerate downward at \(-g\), we’re ignoring air resistance. For things which are reasonably dense and not falling very far, this is a good approximation. For things like feathers, paper, and soap bubbles, it’s a bad approximation. We’ll explain why this is a problem when we discuss the relationship between force and acceleration.

The only other change we’ll make to the kinematic equations is to replace \( x \) with a \( y \). This is just a convention – it’s something people do, but it has absolutely no physical significance. The formula works the same way no matter what variable you decide to use, but when you’re working in more than one dimension, as we soon will be, it’s convenient to call one direction \( y \) and the other one \( x \).

To compare a couple of situations, let’s go back to the very tall bridge we used as an example not long ago. We know that if we drop something (and ignore the air), it will take 4 seconds to hit the water from our 78.4 m bridge. What if we give the rock an initial velocity? If it’s up, it should take more than 4 seconds to hit the water, but if we throw it down, it should take less. Assume you can throw the stone at 30 m/s (about 67 mph). First, throw it up in the air. Up is the positive direction, so \( v_0 \) is +30 m/s. We have another choice – we can either let \( x = -78.4 \) m and \( x_0 = 0 \) m, or we can let \( x = 0 \) m and \( x_0 = 78.4 \) m. Does it make a difference? Not really. If you look at the third kinematic equation, we could take the \( x_0 \) over to the left hand side and have \((x-x_0)\) over there. If we do that, either choice above gives us -78.4 m on the left side (it’ll be negative because we already chose down as the negative direction, and the change in position, or \( x-x_0 \), is definitely down). Now we have:

\[-78.4 = (30m/s)t + \frac{1}{2}(-9.8 m/s^2)t^2\]

We now have a quadratic equation, which you’ve probably seen before & solved before. If we have an equation of this form,

\[ at^2 + bt + c = 0 \]
then the solutions for \( t \) are:

\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

In our case, our two choices for solutions are \( t = -1.98 \) sec. or +8.1 sec. We clearly want the positive root here (although the negative root has some physical significance, which we’ll see in a minute). How fast is it rock going when it hits? Well, the rock will climb up until its velocity is exhausted \((v = 0)\). The height the rock reaches before stopping can be found by using equation 4:

\[
(0 \text{ m/s})^2 - (30 \text{ m/s})^2 = 2(-9.8 \text{ m/s}^2)(y - 78.4 \text{ m})
\]

\( y \) is found to be 124.3 m. As a check on our math, the time it will take to reach this new altitude is just

\[
v - v_0 = at \quad 0 \text{ m/s} - 30 \text{ m/s} = (-9.8 \text{ m/s}^2)t
\]

This tells us that the rock will have reached its 124.3 m height after 3.06 seconds. The rock is momentarily stopped at that altitude, and begins to fall. The time it takes for this part of the fall is simpler: we use equation 3, and set the initial velocity \( v_0 = 0 \text{ m/s} \). We get

\[
-124.3m = \frac{1}{2}(-9.8 \text{ m/s}^2)t^2
\]

so \( t = 5.04 \) seconds. You’ll notice that the time it takes to reach its maximum height plus the time to hit the water from that point is 8.1 seconds, exactly what we calculated with the quadratic formula. Finally, how fast is the rock going when it hits?

Back to equation 4:

\[
v^2 - v_0^2 = 2\left(-9.8 \text{ m/s}^2\right)(-124.3 \text{ m})
\]

Remembering that our initial velocity \((v_0)\) was 0, this gives 49.4 m/s for the speed just before striking the water. Now we can say what the negative root of the quadratic equation actually meant – if the rock is supposed to be moving upwards at 30 m/s at time \( t = 0 \) and a height of 78.4 m above the water, it could have been fired from the water’s surface at 49.4 m/s at time \( t = -1.98 \) seconds.

Check for yourself that, if we had thrown the rock down with the same velocity, it would have reached the water in 1.98 seconds (does that time sound familiar?).
Components of Motion and Vector Addition

(This is a summary of the information you can find in the page about trigonometry & triangles, located at [http://www.chemistry.armstrong.edu/baird/trig.tri.pdf](http://www.chemistry.armstrong.edu/baird/trig.tri.pdf).) So far, we’ve looked at objects moving in one dimension – either up and down (usually called $y$ in 2-D) or left-right ($x$). These cases are a little more specialized than what generally happens when something is moving and has motion in both the $x$ and $y$ directions. When both of these motions are happening at once, it’s time to start using vectors. In some cases, we’ll use the individual components (for example, we may say something is moving 4 m/s E and 3 m/s N) and at other times we’ll talk about the overall magnitude and angular direction of the vector. See below for both methods together.

![Diagram showing vector addition](image)

The vector here is $\mathbf{v}$ and the components of $\mathbf{v}$ are called $v_x$ and $v_y$. Notice that the components are not in boldface – the components are not vectors. The magnitude of the velocity vector is found by treating the components like legs of a right triangle and using the Pythagorean theorem. The magnitude of it is then

$$
\mathbf{v} = \sqrt{v_x^2 + v_y^2}
$$

For the example above, that would give us 5 m/s for the total velocity. The angle $\theta$ can be found by using any of the trig functions on the three legs of the triangle (made from the two components and the final vector). We could use Sine or Cosine, since we know all three sides of the triangle now, but it’s more common to use the Tangent. Tangent is the only trig function that would connect the two components and the angle directly. We find the angle by

$$
Tan(\theta) = \frac{v_y}{v_x} \quad \text{or} \quad \theta = Tan^{-1}\left(\frac{v_y}{v_x}\right)
$$

In our case we get 36.9°. Something important to notice is that we chose to measure the angle from the $x$ axis. We could just as easily measure it from the $y$ axis, or the negative $x$ axis, or wherever. If we do that, though, our formula above for the angle will be altered.
If we’re combining vectors, we can do it one of two ways. The most mathematically accurate way to do it is to add the components. If we take our velocity vector above, and add another vector to it (let’s say it has a \( y \)-component of 10 m/s and an \( x \) component of \(-2 \) m/s), we need to add the components individually and then recombine to form the final vector. The sum of the \( x \) components would be \( 4 \text{ m/s} + (-2 \text{ m/s}) = 2 \text{ m/s} \). The \( y \) components add to give \( 3 \text{ m/s} + 10 \text{ m/s} = 13 \text{ m/s} \). What’s the length of this new vector? Square each final component and add them, taking the square root of their sum: \( \sqrt{(2 \text{ m/s})^2 + (13 \text{ m/s})^2} = \sqrt{173 \text{ m}^2/\text{s}^2} = 13.15 \text{ m/s} \). We can find the angle by the method above, and we get 81.25\(^\circ\). This makes sense, because we’re measuring the angle from the \( x \) axis, and we have a vector that is very much larger in the \( y \) direction than the \( x \) direction, so it shouldn’t be a surprise to see an angle close to 90\(^\circ\).

The other way we can add vectors is called the graphical method (or polygon method), where we put the vectors together, head to tail, and see where they go. In the example below, we start with our two vectors mentioned above and then move the tail of one to the head of the other (keeping it in the same orientation – we can’t rotate it at all, or we’ll get garbage). The final vector, also called the **resultant**, is the arrow from the first tail to the final head:

![Graphical Vector Addition](image)

We can add more than two vectors the same way – we just keep putting tails at the heads of other vectors, and when we’re done, we connect the first tail to the last head. Subtracting vectors is no harder – for the component method, we just subtract the components. For the graphical method, we just add the negative of the vector, which is the same as subtracting it.

A concept which is also useful is that of a **unit vector**. This is just a vector with magnitude equal to one in whatever units are being used. We can take advantage of this to give a shorthand way of talking about vectors. Instead of saying a vector has an \( x \) component of 7 meters and a \( y \) component of \(-3 \) meters, we could also write this vector (call it \( \mathbf{r} \)) as \( \mathbf{r} = 7 \mathbf{x} -3 \mathbf{y} \). These are sometimes written with “hats” over the letters instead of arrows: \( \vec{r} = (7 \text{ m})\hat{x} + (3 \text{ m})(-\hat{y}) \).

**Other Vector Operations**

There are two other common things we can do with pairs of vectors. We can look at their similarities (the amount of overlap between them) by taking the **dot product** of the two vectors. To do this, we combine the \( x \), \( y \), and \( z \) components of each to form a scalar, like this:
We can also find the dot product by multiplying the lengths of the two vectors together, and then multiplying by the cosine of the angle between them:

\[ a \cdot b = \|a\|\|b\|\cos \theta \]

As this form makes clear, if two vectors are perpendicular (so \( \theta = 90^\circ \)), their dot product will be zero. This means there is no overlap between the two; the dot product of a North-pointing unit vector and an East-pointing unit vector is zero, since North has no Easterly component, and vice versa. If the two vectors are parallel (\( \theta = 0^\circ \)), the overlap is total and the dot product is just the product of their lengths.

We can also combine two vectors to form a third vector. This is known as a **cross product** or **vector product**. Your book has an appendix with the component form of the cross product in it, but there is another, much simpler form:

\[ \vec{c} = a \times b = \|a\|\|b\|\sin \theta \]

The direction of the new vector \( \vec{c} \) is chosen so that it is perpendicular to the other two. You can use the **right hand rule** to clear this up: point the fingers of your right hand in the direction of \( a \), and then curl them into the direction of \( b \). Your thumb will be pointing in the direction of \( c \). In contrast to the dot product, you get the largest value of the cross product when the two vectors are perpendicular, and you get zero if they are parallel.

**Projectile Motion**

When an object is moving under the influence of gravity, we call the resulting motion **projectile motion** (because it describes anything thrown or shot, as long as it’s not powered, like a rock, baseball, or cannonball). At first, this sounds like something that would be very complicated – a baseball is thrown into the air at some angle (measured relative to the horizontal, usually) and some speed, and we need to predict things like where it will land, how fast it will be moving when it hits, and how long it will take to get there. The part that saves us lots of work is the fact that we can separate this two-dimensional motion into two one-dimensional motions. Gravity acts only on the \( y \)-component of velocity and position – the velocity in the \( x \)-direction is the same from the moment the pitcher releases it until it hits the ground. (This is making the approximation that the air does not affect the path of the ball at all. This is obviously a very poor approximation for a ping-pong ball, but it’s a pretty good one for most other projectiles.)

Stop for just a minute & re-read that. It’s hugely important, and makes our calculations much easier. Since we can break this otherwise-complicated motion into two separate one-dimensional problems, we can use the 4 (3, really) kinematic equations we’ve been using to solve each piece
separately. One painful problem becomes 2 simple problems. This also illustrates why we’ve spent some time talking about vectors and components. Breaking the 2-D motion into two problems means we have to get the **components** of initial velocity. We do that with trig. Once we’ve solved our two problems, we have to join the results back together, which means using the Pythagorean theorem and more trig.

The simplest case is when a projectile is fired horizontally (so that the angle involved is 0°). Assume that the velocity of the object is \( v = (25 \text{ m/s}) \) in the \( x \)-direction. We also have to specify height from the ground – it’s what will determine when the projectile hits the ground (we’ve already said the speed in the \( x \)-direction has nothing to do with time in the air). Let’s say that the height from the ground is 1 m (of course we’ll also say that the ground is level). How long will it be in the air? We go back to our 4 kinematic equations, and we need the third one. We’ll replace \( x \)-’s with \( y \)-’s since we’re looking at the effects of an acceleration in the \(-y\) direction. Also, since the initial velocity had no \( y \)-component, we can drop the term involving \( v_0 t \). We get this:

\[
0 = 1m + \frac{1}{2}(-9.8 \text{ m/s}^2) t^2 \quad t = 0.452 \text{ s}
\]

If the ball is in the air for 0.452 seconds, and it travels 25 m/s in the \( x \)-direction, we get a range of 11.3 m. What will the new velocity be just before it hits? We still have the same 25 m/s in the \( x \)-direction, but in the \( y \)-direction, we’ll have (using the second kinematic equation) \( v = (-9.8 \text{ m/s}^2)(0.452 \text{ s}) = -4.43 \text{ m/s} \). The total velocity is then \( \sqrt{(-4.43 \text{ m/s})^2 + (25 \text{ m/s})^2} = 25.4 \text{ m/s} \). What direction? Using \( \theta = tan^{-1}(v_y/v_x) \), we get an angle of -10°. This makes sense, because we can see from the magnitudes of the components that the \( x \)-component of velocity is still much larger than the \( y \)-component, so we expect an angle much closer to 0° than 90°. Also, the angle should be negative, since the \( y \)-component of velocity is negative.

Things aren’t much more difficult when the projectile has components of velocity in both the \( x \) and \( y \) directions. The main change is that our time of fall equation (3rd kinematic equation) now has the \( v_0 t \) term that we left out in our first example. The effect of this is that we now have to solve the quadratic equation, just like we did in Chapter 2. Once the time is known, range is found by multiplying it by the \( x \) component of the velocity.

Sometimes, we can use shortcuts to find these answers. The curve made by a projectile is called a **parabola**. If a projectile lands at the same elevation from which it was fired, the parabola will be symmetric. We can use that symmetry to find the total time the projectile spends in the air. The time to go from the initial elevation to the maximum height is the same as the time needed for gravity to change the \( y \) component of velocity to zero. That time (from kinematic equation 2) is \( 0 = v_0 \sin \theta + (-9.8 \text{ m/s}^2) t \). The total time in the air is then \( 2v_0 \sin \theta / g \). Since the range is just the \( x \) component of velocity multiplied by the time in the air, we get

\[
R = v_x t = \left( v_0 \cos \theta \right) \left( \frac{2v_0 \sin \theta}{g} \right) = \frac{v_0^2 \sin(2\theta)}{g}
\]

As with every formula, you shouldn’t use it if you don’t understand where it comes from. Knowing the origin of this one tells us that if the firing and landing points are at different
elevations (firing from a building to the street below), this range formula will not give us the correct answer. Incidentally, you’ll notice that this formula says the maximum range occurs when the projection angle is 45°. This is because of the tradeoff between small angles, which have large $x$ components of velocity but which don’t spend much time in the air, and large angles, which spend longer in the air, but don’t travel too far because they have only small $x$ components of velocity. The plot below shows this dependence on angle.

![Graph of Projectile Motion](image)

Relative Velocity

When we talk about velocities, the important thing is really relative velocity. In fact, there is no such thing as an absolute velocity – all velocities are measured relative to something else. For example, if you and a friend are playing catch on an airplane which is moving at 600 km/hr relative to the ground, that doesn’t mean that you have to be able to catch a ball thrown at 650 km/hr. The important thing is that the velocity of the ball relative to you is probably something like 50 km/hr. If someone on the ground could see it, they would say that its velocity is 550 km/hr when you’re throwing towards the rear of the plane, and 650 km/hr when you’re throwing towards the front. If the plane had an open window and you pitched the ball to someone on the ground, the speed relative to the ground is suddenly very important to that person!

In general, relative velocities are found by adding or subtracting the two velocities involved. In the case of vectors, subtraction is just the addition of the negative of the vector. As mentioned above, a plane’s speed is typically measured against both the air and the ground. The airspeed determines whether the plane will fly or drop out of the air. The ground speed determines how quickly you get to another place on the Earth’s surface. This is why planes typically take off into the wind – if they’re pointed into the wind, they already have wind flowing over the wings at (for
example) 30 km/hr, so they only need to be moving at 100 km/hr (wild guess) relative to the ground to take off (if we assume that you need air moving over the wings at 130 km/hr to take off). Trying to take off with the wind at your back would mean you would need a ground speed of 160 km/hr before the wind over your wings was moving at 130 km/hr.

**Force, Net Force, and Newton’s Laws of Motion**

You probably already feel familiar with the idea of force. Basically, a force is something that acts to change the motion of a body. Force is a vector – it has both magnitude and direction. The important part is generally the net force. The net force is the vector sum of all the forces acting on a body. For example, in a tug-of-war, if the two sides are evenly balanced, the rope doesn’t move. If one side is slightly stronger than the other, the net force may be small, so the rope will slowly move towards the stronger side. The first of Newton’s three laws of motion talks about this: *In the absence of external forces, a body at rest tends to stay at rest, and a body in motion tends to stay in motion.* This sounds somewhat obvious to us now, but the part about a body in motion staying in motion would have seemed strange to people several centuries ago. Watching things on Earth suggests that the natural state of all objects is to be at rest. In fact, it’s just as natural for something to continue moving at a constant velocity, but we usually see the effects of frictional forces on moving objects. Those external forces gradually slow objects until they stop. In space, where there is no air resistance or friction, objects keep moving at the same speed forever unless they fall under the gravitational influence of some other body.

Friction is an example of a contact force, because the two bodies involved are actually in contact. If you push a cart across the floor, that’s also a contact force. The other type of force is called an action-at-a-distance force. Examples of this are gravity and electromagnetism. These forces can influence the motion of bodies when they are far apart (these two actually have no limit on their range – the force gets weaker with increasing distance, but the gravitational force between two things is not zero for any finite distance).

The resistance to being accelerated (or decelerated) is called inertia or mass. This is also part of Newton’s second law of motion. This law boils down to $F = ma$. In this formula, $F$ is the applied external force, $m$ is the mass of the body on which the force acts, and $a$ is the acceleration of the body. (When we look at rockets, we’ll see that this is a slightly simplified version of what Newton’s 2nd law really says, but this is good enough for now). This tells us that mass is really what resists motion. Larger masses are harder to accelerate, which seems obvious – pushing a wheelbarrow will make it accelerate much more quickly than exerting the same force on a car. The SI unit of force is called the **Newton** (N). We can look at the dimensions of acceleration times mass to see that a Newton is $(1 \text{ kg}) \times (1 \text{ m/s}^2) = 1 \text{ kg} \cdot \text{m/s}^2 = 1 \text{ N}$. For comparison, a pound is the English unit of force which is equal to 4.45 N.

This brings up the distinction between weight and mass. Mass is fundamental and unchanging, and it essentially measures the amount of matter inside something. You can think of it as a rough count of the number of protons, neutrons, and electrons (pieces of atoms) in an object. A 1-kg mass, for example, is the same on the Earth, the Moon, the Sun, in space, or anywhere else. It may be weightless in space, but it’s not massless. For example, if the astronauts in the space shuttle take a baseball bat up with them, it will be weightless once they’re in orbit. If it were also massless, they could hit each other with it and it wouldn’t hurt!
Weight is the pull of gravity acting on a mass (we’re usually talking about the pull of Earth’s gravity on a mass). We can then say that your weight would be only 1/6\textsuperscript{th} as much on the Moon as it is on Earth, even though your mass would be unchanged. For that reason, we can’t really convert from pounds to kilograms – pounds are force, and kilograms are mass. Of course, we know that in almost every case, we’re talking about objects which are on the Earth, and therefore their masses and weights are related by the formula below (which stems directly from Newton’s 2\textsuperscript{nd} law:

$$F = ma \quad \text{or} \quad W = mg$$

From this, we can see that one kilogram multiplied by one “g” will give \((1 \text{ kg}) \times (9.8 \text{ m/s}^2) = 9.8 \text{ N}\). We already know that one pound is 4.45 N, so we can see that one kilogram on Earth’s surface will be pulled down with a force of 2.2 pounds (= 9.8 N). This is an incredibly important result – when we want to know how anything moves, what we will always do is sum up all of the forces. \textbf{If their sum is not zero, the object will accelerate.} This is another point to go back & read again.

It doesn’t seem like there is any content in this so far, but try this example. Let’s say a weightlifter can exert an upward force of 1000 N. If a mass of 125 kg is dropped on him (slowly, of course!), what will happen? Will he be able to lift it, or will it hit the ground? How long will it take to go up or down? We can find out using Newton’s 2\textsuperscript{nd} law. The weight of 125 kg on the Earth’s surface is \(W = mg\) or \(W = (125 \text{ kg}) \times (9.8 \text{ m/s}^2) = 1,225 \text{ N}\). This is too heavy for the weightlifter, so it will hit the ground (or his chest). How quickly? Will it accelerate downward at \(g\)? We now look at the net force on the barbell. There’s a downward force of 1,225 N, but an upward force of 1,000 N is also acting on it. The net force (if we make downward negative) is \(-1225 \text{ N} + 1000 \text{ N} = -225 \text{ N}\) (negative, so it’s downward). We use Newton’s 2\textsuperscript{nd} law to find the acceleration of this barbell. The acceleration is the net force divided by the mass:

\[a = \frac{F}{m} = \frac{-225 \text{ N}}{125 \text{ kg}} = -1.8 \text{ m/s}^2.\]

How long will it take for this to hit the weightlifter’s chest? Assume it will drop 0.6 m from the point of handoff until contact with the chest. We know distance, initial velocity (0 m/s if it was slowly handed to the person), and acceleration. How long it takes is then found using our 3\textsuperscript{rd} kinematic equation again.

\[0 = 0.6 + (1/2) \times (-1.8 \text{ m/s}^2) \times t^2.\]

This gives 0.82 s. Quick, but not nearly as quick as a pure fall (which would have had an acceleration of “g”).

\textbf{Newton’s Third Law}

Newton’s 3\textsuperscript{rd} law says that for each action, there is a reaction. In other words, when one body (the Earth, for example) exerts a force on another body (you), the second body exerts an equal and oppositely directed force on the first body. You pull on the Earth with the same force that it pulls on you. Newton’s 2\textsuperscript{nd} law explains why that force alters your motion so much more than it alters Earth’s. This concept confuses people, because they think that if the forces are equal and opposite, there should be no motion. The crucial part that is usually missed is that these two forces act on different objects. Go back to the idea of a tug-of-war. This time, tie one end of the rope to a car that’s stuck. There may be no motion initially, but the force of the people on the rope is the same magnitude as the force of the rope on the people. Once the motion starts, the force of the people on the rope (and the car) is still balanced by an opposing force.
Even when we’re talking about things just sitting around on the Earth, there is still a force which balances gravity (we know there is, because if the only force was gravity, Newton’s 2nd law would force things to start accelerating at “g”). This force is called the normal force (normal meaning that it is directed normal (perpendicular) to the surface it’s resting on). Objects exert a force due to their weight on the surface they rest on, and the surface responds with a normal force.

It’s important to keep in mind that Newton’s laws apply in inertial (non-accelerating and non-rotating) frames.

Free Body Diagrams

When analyzing a problem in mechanics, the first step is always to make a free body diagram, which shows all forces acting on all bodies. The resultant of all of the forces on a body (together with the mass of the body) determines the motion of that body. An example of this (found in the book) is called the Atwood machine. This consists of two masses suspended from either side of a pulley. The only function of the pulley is to change the direction of the forces – there’s no way it can change the magnitude of the forces. We’ll usually ignore complications (like any forces besides gravity and tension in the rope, or the mass of the rope itself) when we look at this machine, although we could include things like friction in the pulley. In this approximation, we have three possible cases which depend on the masses involved. If \( m_1 > m_2 \), \( m_1 \) will move down and \( m_2 \) will move up. If \( m_2 \) is larger, the reverse will happen. Finally, the two masses could be equal, and we would then expect no motion at all. See the diagram below for one possibility:

Let’s see how to find the value of the acceleration (the same for each mass since they’re connected) and the tension (also the same throughout the rope). We can use these two facts to figure out what’s happening. From the drawing, we see that there are exactly 2 forces on each mass: the weight of the mass, and the tension in the rope (which we’ve said is the same for both masses). The sum of the two forces on \( m_1 \) will be equal to \( m_1 \) times its acceleration, and we can write a similar (but not quite identical) formula for \( m_2 \). Again, the accelerations will also be equal, so let’s see what we get:
\[ m_1g - T = m_1a_1 \quad T - m_2g = m_2a_2 \]

Notice from the drawing that \(a_1\) and \(a_2\) must be oppositely directed. We can then call one of the \(a\) and the other \(-a\). Because \(m_1\) is larger, we know that it will accelerate towards the ground, which is the negative direction. For that reason, we’ll set \(a = a_2\) and \(-a = a_1\). Now we get

\[ T - m_1g = -m_1a \quad T - m_2g = m_2a \]

Let’s get the tension term by itself in each case. When we do that, whatever is on the other side of each equation will equal \(T\), and since the \(T\)'s are equal, the other side of the equation for \(m_1\) will equal the other side of the equation for \(m_2\):

\[ T = m_1g - m_1a \quad T = m_2g + m_2a \]

\[ m_1(g - a) = m_2(g + a) \]

Solve this for \(a\) (again, it’s the same for each mass) and we get:

\[ a = \frac{(m_1 - m_2)g}{(m_1 + m_2)} \]

It’s important to notice that, in this case, we looked at each mass separately; we found the forces acting on \(m_1\) to be its weight and tension. We applied a similar procedure to \(m_2\). One way to take a shortcut here is to look at the net forces involved. There’s a downward force of \(m_1g\) on the first mass, but there’s also an upward force of \(m_2g\) on the first mass. This upward force is really the ordinary downward force of gravity on \(m_2\) which has had its direction changed by the pulley. The net force on the masses is therefore \((m_1g - m_2g)\), but this force acts on the total mass of the system, which is \((m_1 + m_2)\). By Newton’s 2nd law, the acceleration is the net force divided by the total mass on which it acts, so we get the result above. In this case, the system is the combination of the blocks and the rope, so tension is an internal force, and therefore doesn’t appear in our equations. Notice that we got the same answer either way; we just have to be clear about what constitutes our system, and therefore what forces are external to it.

You can also find the tension by realizing it would be equal to the smaller of the masses multiplied by gravity if the machine were motionless. Since it’s actually accelerating, we have to multiply the smaller mass by \((a + g)\) to get the tension. We should also be able to multiply the larger mass by \((g - a)\) and get the same number for tension.

The general idea in solving these problems is to sketch the setup and then draw in the force vectors on each moving point. Something else to consider is the fact that you can always choose your own orientation for your \(x\) & \(y\) axes. For example, if we’re looking at a mass on an inclined
plane, we are free to keep the same orientation for our axes that we used in projectile problems (i.e., up/down is \( y \) and left/right is \( x \)). We can do that, but it’s not smart. We should take advantage of the problem’s physical characteristics to orient things so that we have to do less work. For the inclined plane, we usually do this by choosing the \( y \) axis as normal to the plane, and the \( x \) axis as up and down the plane.

**Equilibrium**

If all of the forces on an object are balanced, so that there is no net force, we say that the object is in **translational equilibrium**. This is distinct from rotational equilibrium, which we’ll discuss later. Equilibrium can be **static** or **dynamic**. Static equilibrium is what we expect from things like a hanging sign. We want the net force, and therefore the net acceleration, to be zero, but we also want the velocity to be zero relative to us. Dynamic equilibrium is illustrated in the example below, where two boats (triangles) are cooperating to tow a water skier.

If each boat pulling the water skier is pulling with 600 N of force, and each is pulling at a 45° angle to the \( x \)-axis, what is the magnitude of the retarding force exerted on the skier by the water if the boats & skier are traveling at a constant velocity? First, we can draw the forces on the skier

In component form, \( F_R \) (the retarding force) is entirely in the \(-x\) direction, and (if there is no net force) the \( x \) components of the other forces balance this. Those components are \( F_1 \cos(45^\circ) \) and \( F_2 \cos(-45^\circ) \). The only \( y \) components involved are from the boats. These must also balance, so we get the equations below:

\[
\sum F_x = F_1 \cos(45^\circ) + F_2 \cos(-45^\circ) - F_R = 0
\]

\[
\sum F_y = F_1 \sin(45^\circ) + F_2 \sin(-45^\circ) = 0
\]

Keep in mind that the two angles involved are not both 45°. The angles must be measured in the same direction for things to make sense. This means we can call the second angle 315° or -45° (check for yourself to see that it makes no difference in the math). Anyway, the formulas above show us that \( F_1 = F_2 \) in magnitude (but we knew that anyway, since the problem told us that each
boat was applying a force of 600 N). When we plug those numbers into the formula for the \( x \) components of the force, we get that \( F_x = (600 \text{ N})(0.707) + (600 \text{ N})(0.707) = 849 \text{ N} \) (in the \(-x\) direction).

### Friction

In real-world situations, frictional forces are frequently important in the analysis of motion. Friction behaves very differently from most of the other forces we’ve looked at so far. A frictional force only acts in response to an applied force, it always resists motion, and it varies in size from zero (if nothing is trying to push the object) to a maximum level we’ll discuss later. The cause of friction between solids in contact is generally accepted to be due to bonding between the microscopically small parts of the surfaces which are actually touching each other. This is a kind of micro-welding between the surfaces, and it takes a force to break those tiny welds. We won’t look much more closely at the causes of friction, because it’s a complicated subject. The types of friction are more important for us. When an object is stationary and you’re trying to move it, you’re working to overcome static friction. If you’re able to get it moving, you still have to exert a force on it to keep it moving – the force you exert now is opposing dynamic friction. Finally, when you’re trying to push a car on a flat road, you’re working against rolling friction. In general, static friction is greater than dynamic, which is greater than rolling. In other words, when you’re moving a refrigerator, it’s easier to keep moving than to get moving. Also, if you put it on a wheeled cart, it’s even easier to move it.

The general form for frictional force in the static case is:

\[
f_s \leq \mu_s \, N
\]

Notice that this formula has the “less than or equal” sign. In other words, the frictional force will be somewhere between 0 and the quantity on the right. It’s easy to see why it varies – assume the quantity on the right is equal to 10 Newtons for some situation. If you are only pushing with a force of 5 N, we know that the object should sit still. If it resisted with a force of 10 N in the opposite direction, it would actually start moving towards your hand if you pushed it lightly! We know that it won’t do that – we may not be able to apply enough force to move it, but we know things don’t come after us if we’re too weak to move them! What do the quantities on the right represent? The \( N \) in the formula is the normal force of the object we’re talking about. The \( \mu_s \) is called the coefficient of static friction. This has to be a dimensionless quantity (we know that, because \( N \) is a force (Newtons) and the frictional force is also in Newtons).

Once the object is in motion, the formula for the dynamic frictional force is:

\[
f_k = \mu_k \, N
\]

The \( k \) is for kinetic friction, another name for dynamic friction. If you apply a force less than the coefficient of static friction multiplied by the normal force, the object you’re pushing will not move. If the force is greater, it will start moving and the equation for kinetic friction will determine the force needed to keep it going. Applying a force greater than \( f_k \) to the object will cause it to accelerate.
Inclined plane problems are favorites for illustrating friction. We know that, as the gravitational force tries to drag the block (or whatever) down the plane, the frictional force tries to prevent the motion. We expect the gravitational force to increase as the angle of the plane increases (a steeper slope will allow gravity to more directly pull on the block). The thing that you might not have expected is that the frictional force will decrease as the tilt angle increases. This is because the normal force is no longer anti-parallel to gravity. \( N \) is equal to \( mg \cos \theta \) on the plane, and that force drops as the angle increases. You're already aware of this if you’ve ever noticed that things don’t tend to sit on walls as well as they do on floors! One of the ways to find the coefficient of static friction involves the plane – since frictional force will be equal to the force trying to drag the block down the plane (until it actually moves), we know that \( f_k = mg \sin \theta \).

We also know that (when it starts to move) it’s equal to \( \mu_s N = \mu_s mg \cos \theta \). Using the equation above, we see that it boils down to \( \mu_s = \tan \theta \). This tells us that an object which only starts sliding at an angle of 45° has a coefficient of static friction of exactly 1. The table in your book shows that, in general, these coefficients are less than 1.

When solving problems relating to friction, the fact that the equation for static friction has a “less than” sign in it is a slight but very important complication. The procedure is this: first, ignore friction and add up the forces involved as if friction did not exist. When you get to the end of this preliminary problem, you need to compare the net force with the maximum force of static friction, which is \( \mu_s N \). If the net force is less than this maximum, **the object does not move**, and the frictional force is exactly equal and opposite to the force calculated in the absence of friction. If the friction-free net force is greater than the maximum frictional force, take the difference between the two and that is the true net force on the object, and it will move. One of the most common mistakes students make when solving problems involving friction is to put in the frictional force (at its maximum) just like any other force and solve away. This mistake will be obvious if, at the end of your problem, you find your mass moving up an inclined plane with friction present. Now take a look at the situation below and see if you know, just by looking, which way the block should move with (and then without) friction:

Should block 1 go up the plane, or down? There’s no way to know without knowing \( m_1, m_2 \), the angle of the inclined plane, and the coefficient of static friction \( \mu_s N \).

**Air resistance** is a different kind of frictional force, and is a velocity-dependent force. The shock absorbers in your car and on self-closing doors also provide velocity-dependent forces – if
you try to open the door very quickly, or if you hit a bump at high speed, the shock absorber will
strongly resist the motion, unless the force is large enough to break it! You can demonstrate this
in a pool. Waving your hand quickly underwater takes much more effort than slowly moving it.
The exact form of the velocity dependence is itself velocity dependent!

Air resistance is important because it’s what makes us think that heavy things fall faster. We
know that all falling objects near Earth’s surface accelerate at \( g \) initially, but we also know that
sheets of paper fall more slowly than lead weights. It’s because the object only accelerates until
the force of air resistance matches the force of gravity. When that happens, the object stops
accelerating and falls at a constant velocity known as its **terminal velocity**. The sheet of paper
has a large area and a shape that tends to catch air very efficiently. These factors combine to give
it a very small terminal velocity, which it reaches very quickly as it flutters to the ground. A
sphere made of lead has an aerodynamic shape and will have a small surface area compared to its
weight (that’s why we say lead is dense). This lead weight will have a very high terminal
velocity and it will have to fall for quite some distance before it reaches that terminal velocity, so
it will make it to the ground much more quickly unless both things are falling in a vacuum.

To calculate the drag force more exactly, we need to look at what factors influence it; the area of
the object moving through the air (or fluid) is clearly important, since carrying a piece of
plywood in the wind is much easier than carrying a stamp in the wind. Also, the force must
depend on velocity, as you can confirm by placing your hand outside of a car window as it
accelerates. The density of the fluid you’re moving through will also matter – it’s much easier to
walk through the air than through chest-deep water. Finally, the shape of the object will matter.
The next time you see a car from the 70’s, notice how square it looks compared to today’s
heavily rounded cars. That’s not just a styling choice. All of these combine in the formula
below:

\[
\vec{D} = \frac{1}{2} C \rho A v^2
\]

If \( D \) is a force, we can see that \( C \) must be dimensionless. Additionally, the area here is the area
perpendicular to the direction of motion (which is in the direction of \( v \)). We can use this
expression for \( D \) to find terminal velocity \( v_t \) by noticing that \( v_t \) must be the point where \( D \) is
balanced the object’s weight. We can write

\[
\vec{D} = \vec{W} \quad \text{so} \quad v_t = \sqrt{\frac{2 m g}{C \rho A}}
\]

For two objects with the same shape/surface texture (so \( C \) is the same) and the same cross-
sectional area moving in the same medium, we find that the heavier of two objects **does** fall
faster! Of course, in a vacuum, all things will fall with the same acceleration.
Uniform Circular Motion & Centripetal Acceleration

A relatively simple kind of circular motion is known as **uniform circular motion**. The motion is uniform in the sense that the object’s *speed* is constant. The object can’t have a constant *velocity*, though, because only things moving in straight lines at constant speeds have constant velocities. When an object moves in a circle, its velocity vector is **tangential** to that circle. We know that objects tend to continue in straight lines unless acted on by a force, so something has to be forcing that object back towards the center of the circle. See below.

The velocity vector would like to carry the rotating blue ball towards the bottom of the page. If the ball is going to remain in a circle, some force must be bending it back towards the person spinning it (smiley face). That force is called the **centripetal force** (meaning “center-seeking”) and produces a **centripetal acceleration** when acting on the moving mass.

Your book shows that the change in the direction of the velocity vector divided by the velocity is the same as the change in position along an arc of the path divided by the radius. We can use these geometrical arguments to prove that the direction of the centripetal acceleration is radially inwards (towards the center) and that its magnitude is:

\[ a_{cent} = \frac{v^2}{r} \]

The acceleration \(a_{cent}\) is the acceleration necessary to **keep** the object moving in a circle. If this is not applied (the string on the yo-yo you’re swinging above your head breaks), the object will fly off tangentially. If we multiply this acceleration by the mass it acts on, we’ll get the **centripetal force** acting on the object. Very much like what we have above, we can write:

\[ F_{Cent} = ma_{cent} = \frac{mv^2}{r} \]

Something worth keeping in mind is the idea of a **centrifugal force**. This is commonly described as “the force that throws you to the side of your car as you go around a curve”. That’s actually not quite right. What “throws” you to the outside of the curve is just Newton’s 1st law.
When you approach a curve, the car and its passengers want to keep going straight. It takes the application of a force to prevent that. The applied force is called the centripetal force, and the friction between the tires and the road provides it. Once you slide across the seat & hit the door, the door will provide the force needed to change your body’s direction. For this reason, centrifugal force is called a fictitious force. It really represents the absence of centripetal force rather than the presence of some other force.

The introduction of the centrifugal force provides us with a way to preserve Newton’s laws in the non-inertial frame of the car. For example, let’s say the driver isn’t too reckless, and that there’s enough friction between your clothes and the car seat to keep you from sliding. You’ll say that your acceleration along the seat is zero. We could then write

\[ f_s - \frac{m v^2}{r} = ma = 0 \quad \text{or} \quad f_s = \frac{m v^2}{r} \]

Notice that the fictitious centrifugal force is directed oppositely to the frictional force, so it comes in with a negative sign, but it’s on the left side of the equals sign since it’s a force. The other way to do this, without the fictitious force, is to say that the only applied force is friction, but that, since you’re moving in a circle, you must be feeling an acceleration of \( m v^2 / r \) instead of zero. The applied force goes on the left, and the \( ma \) goes on the right. We get
\[ f_s = \frac{m v^2}{r} \]

The result is the same. In other words, we have two choices: we can say movement with our rotating frame gives an acceleration of zero, but centrifugal force exists. Alternately, we can recognize that movement in a circle necessarily requires an acceleration of \( v^2/r \), and we have no centrifugal force.