(Notes follow and parts taken from sources in bibliography)

**Inductors**

As the function of a resistor is to oppose the flow of current, there are devices called *inductors* whose function is to oppose *changes* in the flow of current. If the current through an inductor changes, the inductor will respond by acting like a voltage source (directed in such a way as to oppose the current change) with a strength given by

\[ V = L \frac{di}{dt} \]

The faster the current changes, the larger the opposing voltage produced by the inductor. We could also integrate this and get an expression for the current as a function of voltage

\[ i(t) = \frac{1}{L} \int_{t_0}^{t} V \, d\tau + i(t_0) \]

Because the formula for the power consumed by a circuit element is \( P = I \, V \), we can calculate the energy stored in an inductor as it goes from zero current to maximum current:

\[ \int P \, dt = \int LI \, dI \Rightarrow U = \int_{0}^{t} LI' \, dI' = \frac{1}{2} LI^2 \]

Notice that we are not watching this inductor over a full cycle of sinusoidally oscillating voltage; it’s just a current ramping up from 0 to some maximum value. The direction of the induced voltage can be determined by realizing that it opposes the change in current.

**Capacitors**

Inductors produce a voltage in proportional to the derivative of current with respect to time, but capacitors can be thought of as producing a voltage in response to the **integral** of current with respect to time. We can write this as

\[ i = C \frac{dV}{dt} \]
or

\[ V(t) = \frac{1}{C} \int_0^t i \, d\tau + V(0) \]

Finally, as we did with the inductor, we can write an expression for the power consumed as the capacitor charges up to \( V \) volts:

\[ \int P \, dt = \int CV \, dV \implies U = \int_0^V CV' \, dV' = \frac{1}{2} CV^2 \]

The purpose of a capacitor is to separate and store charge. The charging and discharging rates may be the same, or they may be very different. Although current does not flow through a capacitor (because it is just two conductors separated by some kind of dielectric), the buildup of positive charge on one plate and negative charge on the other looks from the outside like a current is flowing (positive charge is leaving the battery headed for the negative plate, and we can think of the negative charge leaving the battery’s negative terminal for the other plate as being identical to positive charges coming from that plate to the battery’s negative terminal, which would look to the battery exactly like a regular current). This pseudo-current is called the displacement current, and it is proportional to the change in voltage over time, as can be seen by looking at the first equation in this section.

**Series and Parallel Combinations**

When multiple inductors are linked in series, they will all carry the same current, and the voltage drop across the full chain will be equal to the individual voltage drops across each inductor. That means that inductances in series work the same way as resistances in series:

\[ L_{\text{series}} = L_1 + L_2 + L_3 + \ldots \]

If the inductors are in parallel, however, it’s the voltage across each that is the same. In this case, the currents add to produce the total current, and the equivalent inductance to this group is

\[ \frac{1}{L_{\text{par}}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots \]

With capacitors, things are reversed. If you think about the capacitors as parallel-plate capacitors, and remember from introductory physics that the capacitance of a parallel-plate capacitor is given by
you can see that capacitors with the same separation and dielectric could be slightly redrawn into a single capacitor. That equivalent capacitor would keep its plate separation and dielectric constant, but its area would be equal to the sum of the areas of all of the other capacitors.

\[ C_{\text{tot}} = C_1 + C_2 + C_3 + \ldots \]

If the capacitors are in series they combine in a very different way. Notice that the right plate of the left capacitor and the left plate of the right capacitor in the drawing below are connected by a wire.

The plates must have charges which are equal in magnitude but opposite in sign since the arrangement inside the dotted circle (plates of different capacitors connected by a wire) would be neutral without a battery. That means that the plates facing them also have charges of the same magnitude and opposite sign. We’ll call it \( Q \) and label them all

Since the voltage is split between the capacitors, the sum of the voltages across each should be the battery’s voltage.

\[ V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_{\text{Total}}} \]
The capacitance of several capacitors in series is then:
\[
\frac{1}{C_{\text{series}}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots
\]

**Mutual Inductance**

A current in a coil causes a magnetic field, and a changing magnetic field causes a current in a coil. What happens if one coil is near another and the first (primary) coil is connected to a generator? The primary coil will develop a magnetic field, which will of course change with time since the voltage of the generator and therefore the current of the coil oscillates in time. As this changing magnetic field encounters the other coil (not connected to a generator, and called the secondary), it will cause an induced current to flow in it.

This effect is known as **mutual induction**. The emf induced in the secondary coil depends on the change in magnetic flux through the secondary as time goes on. The magnetic flux in the secondary depends on its area, its number of loops, and the magnetic field (produced by the primary). The primary’s magnetic field is proportional to the current through it, so we can say the total flux through the secondary coil \(N_s \Phi_s\) is proportional to \(I_p\). If the two things are proportional, we can include a constant and set them equal to each other. The constant will be called \(M\), for mutual inductance:

\[
N_s \Phi_s = M I_p \quad \text{so} \quad M = \frac{N_s \Phi_s}{I_p}
\]

which we can rewrite as

\[
V_s = -M \frac{dI_p}{dt}
\]

Keep in mind that \(M\) also depends critically on the arrangement of the coils, so it is a geometric factor as well.

The direction of self-induced voltage has already been discussed, but since the direction of mutually-induced voltage depends on the arrangement of the coils (which is not typically shown in a circuit diagram), we have to have a way to indicate the direction of the voltage induced in this case.

To do this, we use the **dot convention** described in your book. One end of each inductor is marked with a dot. If you measure the voltage across each inductor from its un-dotted end to its dotted end, you’ll see that the voltages oscillate in phase with one
another. As your book states it, "When the reference direction for a current enters the
dotted terminal of a coil, the reference polarity of the voltage that it induces in the other
coil is positive at its dotted terminal". In other words, if the self-induced voltage in one
coil is higher at its dotted end, the voltage produced by mutual induction in the other coil
will also be higher at its dotted end.

When you can see the coils, you can figure out the dotting if it is not present. Using the
right-hand rule, if the fingers in your right hand curl in the same direction that the current
in one of the coils is moving, your thumb will point in the direction of the magnetic flux.
Put a dot on the terminal of the first coil where the current enters. Now, pick one of the
terminals of the other coil and imagine a current entering it. Find the direction of the flux,
again using the right-hand rule. If the direction is the same as that from the flux due to
the first coil, put the other dot by the terminal through which the current entered the
second coil. If not, the dot goes at the other terminal of the second coil.

Mutual inductance combines the effect of the individual self-inductances of the coils
involved with a geometric factor which describes their degree of flux overlap. This factor
is known as the coefficient of coupling (represented by k) and it ranges from 0 for
coils with no shared flux to 1 if two coils could completely share the same flux. This
concept means that we can write the mutual inductance $M$ of two coils with self-
inductances $L_1$ and $L_2$ and coupling coefficient $k$ as

$$M = k \sqrt{L_1 L_2}$$

We will restrict ourselves to linear materials (meaning the flux is linearly related to the
coil current and the number of turns). This rules out iron and other ferromagnets, but we
can still do a significant amount within these restrictions. If we have only two coils,
notice that the equation above would be symmetric for the mutual inductance in either
coil due to the other (i.e., $M_{21} = M_{12} = M$).

If we want to examine the energy stored in two coils which are linearly coupled, we will
find the individual energies of the coils, as expected, but we will also get a term for the
energy stored by the mutual inductance:

$$U = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

As shown in your book, this extra term arises from the voltage induced in coil 1 (for
example) as current $i_2$ goes from 0 to $I_2$. The voltage multiplied by the current and then
integrated over time gives the energy. This can be written as

$$\int p \, dt = \int_0^{I_2} I_1 M \, di_2 = M I_1 I_2$$
**Natural Response of RL Circuits**

Because a perfect inductor only resists changes in current, it will act like an ordinary wire (short circuit) in a DC circuit after the current has reached its steady-state value. If the DC source is disconnected, the inductor will begin to release its energy in the form of a potential across the inductor which will drive a current. If we assume the part of the circuit containing the inductor also contains a resistor, we can see that the energy stored in the inductor will eventually be dissipated as heat in the resistor.

The current will move through the inductor in the same direction as the original current since the inductor is resisting the change by trying to keep the current the same. The opposing voltage is strongest where the rate of change is strongest, which is at the instant of disconnection. The inductor can’t completely replace the current, so its rate of change decreases with time, meaning the current slowly dies and its rate of change drops off. Once the voltage source is no longer in the circuit, Kirchoff’s laws give us

\[
L \frac{dI}{dt} + IR = 0
\]

This differential equation can be quickly solved to give

\[
I(t) = I_0 e^{-(R/L)t}
\]

where \(i_0\) is the current in the inductor just before disconnection at time \(t=0\). The voltage across the resistance \(R\) would of course be

\[
V(t) = I_0 R e^{-(R/L)t}
\]

Since the power dissipated in the resistor is \(P = I V\), the energy lost there after time \(t\) will be

\[
\int_0^t I_0^2 R e^{-2(R/L)\tau} d\tau = \frac{L I_0^2}{2} \left( 1 - e^{-2(R/L)t} \right)
\]

As time goes on, the energy remaining in the inductor goes to zero.

The value \(R/L\) provides a characteristic time for the decay (or buildup) of current in this circuit, and that time is known as the inductive time constant, usually represented by \(\tau_L\). After a time \(\tau_L\), the current will have reached about 63% of its final value. After every
additional time interval $\tau_L$, it will progress 63% of the remaining way to its final value, so that after 10 $\tau_L$, we have reached more than 99.99% of the way to its final value.

**Natural Response of RC Circuits**

Circuits consisting of a resistor and a charged capacitor have quite a bit in common with RL circuits. Since the charge on a capacitor and the voltage between its plates satisfies the equation $Q = CV$, the current in the circuit will depend on the time derivative of the capacitor’s voltage. We can use KCL to see that, in the simplest RC circuit, we must have

$$C \frac{dV}{dt} + \frac{V}{R} = 0$$

The voltage across the capacitor’s plates will then change with time as

$$V(t) = V_0 e^{-t/RC}$$

We can use this to find the current, power, and energy dissipated in this circuit as

$$I(t) = I_0 e^{-t/RC} = \frac{V_0}{R} e^{-t/RC}$$

$$P = \frac{V_0^2}{R} e^{-2t/RC}$$

$$U = \frac{1}{2} CV_0^2 \left(1 - e^{-2t/RC}\right)$$

We can again define a time constant associated with this circuit $\tau_C = RC$.

**Step Responses of RL and RC Circuits**

When current or voltage is applied suddenly to RL or RC circuits, we can then investigate the step response of the circuit. A series circuit consisting of a battery, inductor, resistor, and switch is shown below.

When the switch is closed, the battery will attempt to establish a current of $V_s/R$ in the circuit. The inductor will try to prevent that, and Kirchoff’s Voltage Law gives us
\[ V_S = I R + L \frac{dI}{dt} \]

We can rearrange this differential equation and solve it for \( I \) as a function of \( t \), which will give us

\[ I(t) = \frac{V_S}{R} + \left( I_0 - \frac{V_S}{R} \right) e^{-\frac{R t}{L}} \]

Inspection of this equation shows that, for an de-energized inductor (i.e., \( I_0 = 0 \)) at \( t = 0 \), the current in the circuit will be zero. As time goes on, the current will asymptotically approach the value we would get if the inductor were replaced by a straight piece of wire.

As to the voltage across the circuit elements, we know that the resistor will have a potential difference between its terminals of \( I R \). Finding the potential from the formula above would be

\[ V_{Res}(t) = V_S + \left( I_0 R - V_S \right) e^{-\frac{R t}{L}} \]

The voltage across the inductor, however, will satisfy \( V = L \frac{dI}{dt} \), giving us

\[ V_{Ind}(t) = (V_S - I_0 R) e^{-\frac{R t}{L}} \]

Setting \( I_0 = 0 \) gives us the familiar formula from introductory physics for the current in the series RL circuit:

\[ I(t) = \frac{V_S}{R} \left( 1 - e^{-\frac{R t}{L}} \right) \]

As in the free response case, we can make analogies between the step response of the RL circuit and that of an RC circuit. We can put the capacitor in series with a voltage source and a resistor or in parallel with a current source (delivering \( I_S \)) and resistor, as your book does. We can use the KCL on one of the nodes joining the current source, resistor and capacitor, which will give us

\[ C \frac{dV_C}{dt} + \frac{V_C}{R} = I_S \]
This differential equation for capacitor voltage is similar to the one obtained for the current in an RL circuit. Solving it will give us

\[ V_C(t) = I_S R + (V_0 - I_S R) e^{-t/RC} \]

Dividing this by the resistance, we can also get

\[ I(t) = \left( I_S - \frac{V_0}{R} \right) e^{-t/RC} \]

The initial value of voltage across the capacitor is \( V_0 \). When this is zero, we again recover an introductory physics equation

\[ V_C(t) = I_S R \left( 1 - e^{-t/RC} \right) \]

**RC & RL Circuits - General Solutions**

You've probably noticed the similarities between the RC & RL circuits and their currents and voltages. In general, we could write the differential equation describing these things as

\[ \frac{dx}{dt} + \frac{x}{\tau} = K \]

where \( x \) represents either current or voltage, \( \tau \) is the appropriate time constant, and \( K \) is another constant that may or may not be zero, depending on the particular problem. The solution for the behavior of \( x \) as a function of time can be written

\[ x(t) = x(t_0) + x_f \left( 1 - e^{-\frac{(t-t_0)}{\tau}} \right) \]

where it is common to set \( t_0 = 0 \). This tells us that just knowing the time constant and the initial and final values of the voltage or current is enough to know how voltage/current changes with time in general. Typically, we'll be looking at voltage across capacitors and current through inductors. Keep in mind that these particular things can only change continuously – we can't have an instantaneous jump in either capacitor voltage or inductor current. For other things, you'll need to think of \( x(t_0) \) as really meaning \( x(t_0 + \text{some infinitesimal time}) \).
Unbounded Response

The same equations that predict voltage and current changes in capacitors and inductors also allow for exponential growth as well as decay. This can happen in circuits with dependent sources since the Thevenin resistance at the terminals of the inductor or capacitor can be negative in this case. That leads to an exponentially growing current or voltage. Of course, this can't go on forever because it's physically impossible. Something will break down at some point to stop this growth.

RLC Circuits

If we combine an inductor, capacitor, and resistor in parallel in a single circuit, the behavior becomes a little more complicated. In a parallel circuit, the voltage across each element will be the same, so we can find that first.

If we find the time derivative of the equation giving us all of the currents in each branch, we can write

\[
\frac{1}{R} \frac{dV}{dt} + \frac{V}{L} + C \frac{d^2V}{dt^2} = 0
\]

The general form of the solution for this would be something like \( V = Ae^{st} \). If we plug this into the equation and simplify the form of it, we get

\[\begin{align*}
As^2 e^{st} + \frac{A}{RC} s e^{st} + \frac{Ae^{st}}{LC} &= 0 \\
\Rightarrow Ae^{st} \left( s^2 + \frac{s}{RC} + \frac{1}{LC} \right) &= 0
\end{align*}\]

The only physically nontrivial solution involves finding the roots of the quadratic equation. We can write them as

\[s_1 = \frac{-1}{2RC} + \sqrt{\left( \frac{1}{2RC} \right)^2 - \frac{1}{LC}} \quad \text{and} \quad s_2 = \frac{-1}{2RC} - \sqrt{\left( \frac{1}{2RC} \right)^2 - \frac{1}{LC}}\]

Because we're looking at a differential equation, any linear combination of solutions is also a solution. If we define a resonant frequency \( \omega_0 \) and a neper frequency \( \alpha \) as
\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \alpha = \frac{1}{2RC} \]

we can rewrite our solutions as

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

The general solution can then be written

\[ V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

The initial conditions of the problem determine \( A_1 \) and \( A_2 \) since they arise from constants of integration.

The three physical circumstances that characterize the solutions are underdamping, which occurs when \( \omega_0^2 > \alpha^2 \), overdamping (when \( \omega_0^2 < \alpha^2 \)), or critical damping if \( \omega_0^2 = \alpha^2 \).

Finding the particular values of \( A_1 \) and \( A_2 \) for a the overdamped case is done by evaluating \( V(t=0^+) \) and d\( V(t=0^+)/dt \) and plugging them in the equation above (and its time derivative). In a parallel circuit, since all voltages are the same, it’s easiest to find the voltage across the capacitor \( V_C(t=0^+) \) since that is also equal to \( V(t=0^+) \). Similarly, we can find d\( V(t=0^+)/dt \) easily by recognizing that it is equal to \( I_C(t=0^+)/C \) and using the fact that the sum of the currents at any node equals zero. The resistor current is found from \( V(t=0^+)/R \) and the current in the inductor is given as \( I_0 \).

A common way to write \( s_1 \) and \( s_2 \) if the circuit is underdamped is

\[ s_1 = -\alpha - j \omega_d \quad s_2 = -\alpha + j \omega_d \quad \Rightarrow \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2} \]

where \( \omega_d \) is referred to as the damped frequency and \( j \) is a common engineering notation for \( i = \sqrt{-1} \). The full solution can then be written

\[ V(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \]

Solving for \( B_1 \) and \( B_2 \) is similar to the procedure for solving for the \( A \) coefficients: \( V(t=0^+) = B_1 \) and d\( V(t=0^+)/dt = I_C(t=0^+)/C = -\alpha B_1 + \omega_d B_2 \). An underdamped circuit will oscillate as it approaches its steady-state configuration. An overdamped circuit will move slowly towards its final value without oscillation. If your car’s shock absorbers are worn out, the car’s vertical motion over bumps will be underdamped – it will keep bouncing after
hitting the bump. If the shock absorbers were too large for the car, they would effectively be fighting the springs as the springs tried to return the car to equilibrium, so the car would be slow to return to level travel.

If $\omega_0^2 = \alpha^2$, the circuit is critically damped and $s_1 = s_2 = -\alpha$. This situation of two identical roots to the quadratic equation is what you would get when studying projectile motion if the projectile’s $y$ component of velocity squared equaled $2g \Delta y$. Instead of two solutions for possible times, there is only one, and it appears twice. What this means for differential equations is that the linear combination of the two solutions becomes

$$V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

As in the previous cases, solving for $D_1$ and $D_2$ involves the use of: $V(t=0^+) = D_2$ and $dV(t=0^+)/dt = V_{C(t=0^+)}/C = D_1 - \alpha D_2$.

**Parallel RLC Step Response**

A parallel RLC circuit suddenly connected to a current source providing a constant current of $I$ (known as the **forcing function**) will satisfy the current conservation equation

$$I_L + I_R + I_C = I$$

which can be rewritten as

$$I_L + \frac{V}{R} + C \frac{dV}{dt} = I$$

If we differentiate the current/voltage law for an inductor, we get

$$\frac{dV}{dt} = L \frac{d^2 I_L}{dt^2}$$

which allows us to write the current conservation equation as

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC} \frac{dI_L}{dt} + \frac{I_L}{LC} = \frac{I}{LC}$$
Note the similarity between this equation and the differential equation for the voltage in the parallel RLC circuit. We can regain that form if we rewrite the current conservation law as

\[ \frac{1}{L} \int_0^t V \, d\tau + \frac{V}{R} + C \frac{dV}{dt} = I \]

Taking the time derivative of this simplifies it to

\[ \frac{1}{RC} \frac{dV}{dt} + \frac{V}{LC} + \frac{d^2V}{dt^2} = 0 \]

We already know the solutions to this equation. They are

\[ V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]
\[ V(t) = B_1 e^{-\alpha t} \cos(\omega_0 t) + B_2 e^{-\alpha t} \sin(\omega_0 t) \]
\[ V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \]

The currents (including the forcing current \( I \)) are found using

\[ I_L(t) = I + A_1' e^{s_1 t} + A_2' e^{s_2 t} \]
\[ I_L(t) = I + B_1' e^{-\alpha t} \cos(\omega_0 t) + B_2' e^{-\alpha t} \sin(\omega_0 t) \]
\[ I_L(t) = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \]

**Natural & Step Responses of Series RLC Circuits**

For the natural response of a series RLC circuit, we can use the KVL around the circuit and get

\[ IR + L \frac{dI}{dt} + \frac{1}{C} \int_0^t I \, d\tau + V_0 = 0 \]

where \( V_0 \) is the initial voltage of the capacitor. Differentiating this with respect to time gives
\[
\frac{d^2 I}{dt^2} + \frac{R}{L} \frac{dI}{dt} + \frac{I}{LC} = 0
\]

We can then write a solution as
\[
s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}
\]

where the new neper frequency \( \alpha \) is
\[
\alpha = \frac{R}{2L}
\]

and \( \omega_0 \) is the same as in the other examples. Finally, we end up with
\[
I(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(overdamped)}
\]
\[
I(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \quad \text{(underdamped)}
\]
\[
I(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \quad \text{(critically damped)}
\]

To find the step response of the series RLC circuit driven by a voltage source of strength \( V \), we again use the KVL, but now eliminate current terms by using
\[
I = C \frac{dV_C}{dt} \quad \Rightarrow \quad \frac{dI}{dt} = C \frac{d^2 V_C}{dt^2}
\]

which gives
\[
\frac{d^2 V_C}{dt^2} + \frac{R}{L} \frac{dV_C}{dt} + \frac{V_C}{LC} = \frac{V}{LC}
\]

Our final solutions are then
\[ V_C(t) = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \quad \text{(overdamped)} \]
\[ V_C(t) = V_f + B_1' e^{-\alpha t} \cos(\omega_d t) + B_2' e^{-\alpha t} \sin(\omega_d t) \quad \text{(underdamped)} \]
\[ V_C(t) = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \quad \text{(critically damped)} \]

where (as you can see by letting \( t \to \infty \)) the final voltage on the capacitor is \( V_f \).

**Sinusoidal Sources**

Circuits powered by current or voltage sources which oscillate sinusoidally are extremely common. We’ve looked at the way resistors, capacitors, and inductors behave with no sources, with step-function sources, and we’ve also looked at long-term (steady-state) DC behavior.

If the source never “settles down” to a constant value, we can expect the same from voltages and currents in the circuit. From studying mechanical oscillators, you probably already know that 1) they have a natural frequency that describes their motion when released from a state of nonzero potential energy and 2) if they are driven by an external and time-varying force, their motion will adopt the same frequency as that of the driving force rather than their natural frequency as time goes on.

Your book writes sinusoidally varying voltages in the form

\[ V(t) = V_m \cos(\omega t + \phi) \]

where \( V_m \) is the amplitude of the voltage, \( \omega \) is the angular frequency of the driving source (generally assumed to be constant) and \( \phi \) is the phase angle which is used to match the expression to the initial conditions (for example, if the voltage is at its minimum value at \( t = 0 \), the phase angle is chosen so that \( \cos(0 + \phi) = -1 \). In this case, \( \phi = 180^\circ \).)

Frequently, we’ll want to know what DC voltage would be most nearly equivalent to a given AC voltage. This is determined by finding the rms, or root-mean-squared value of the AC signal. To do this, we just square the expression above for the voltage, integrate it over one full period, divide by the length of that period, and take the square root of the result. We can write

\[
V_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) dt}
\]

The integral of \( \cos^2 \) over one period is just \( \frac{1}{2} \), so we could write
In this way, we can see that a resistor connected across an alternating potential difference with $V_{\text{rms}} = 20$ volts (for example) will dissipate the same power as it would if connected to a 20 V battery. Similarly, we define an average or effective current as

$$I_{\text{rms}} = \frac{I_m}{\sqrt{2}}$$

If we examine the behavior of a series RL circuit which has no current flowing in it at time $t=0$ (the instant a switch closes) and ask what happens after that time, we will still use Kirchoff’s Voltage Law to get

$$L \frac{dI}{dt} + IR = V_m \cos(\omega t + \phi)$$

which has, as a general solution,

$$I(t) = \frac{-V_m}{\sqrt{R^2 + (\omega L)^2}} \cos(\phi - \theta) e^{-Rt/L} + \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t + \phi - \theta)$$

In this expression, $\theta$ is defined by $\tan \theta = \omega L / R$. The behavior of this solution (consisting of a transient term added to a steady-state term) for a few different values of $R$, $L$, and $\phi$ is shown below. For all plots, $V_m = 1$ volt.

In the first example below, we have $R = 100 \ \Omega$, $L = 5$ H, $\omega = 20$ rad/s, and $\phi = -\pi/2$. The two solutions are similar in magnitude for the first 0.1 second or so, but the transient term dies out very shortly after that.
In the next pair of images, we have changed the phase from $\phi = -\pi/2$ to $\phi = \pi/2$, which only changes our starting point.

Notice that in the third pair of figures (R = 10,000 $\Omega$, L = 0.005 H, $\omega = 20$ rad/s and $\phi = \pi/2$) L is so small and R is so large that the transient solution is essentially negligible compared to the steady-state solution. This means that the exponential damping factor in the first part of the general solution for the current very quickly extinguishes the transient response. Also, the larger resistance more than makes up for the smaller inductance when looking at the magnitude of the steady-state current, as it is much smaller now than in the previous examples.

**Phasors and Complex Numbers**

We’ve seen that the voltage across an inductor depends on the time derivative of the current through it, and the voltage across a capacitor can be thought of as the time integral of the current to it. If the voltage or current driving a circuit changes sinusoidally, the time derivative and time integral of that voltage or current will be out of phase with the driving force. The voltage and current in a resistor are always in phase (since $V = I R$ has no time derivatives or integrals).

When resistors and capacitors and/or inductors are all present in a circuit, the way they combine to affect the current can be tricky to determine. The capacitor and inductor are
known as reactive elements, while the resistor is (surprise) a resistive circuit element. Finding the reactive and resistive contributions to a circuit is very similar to the process of two-dimensional vector addition, where reactance is on one axis and resistance is on the other. We are used to writing things like $6i - 5j$ when we mean the vector formed by moving 6 units in the positive $x$ direction and 5 units in the negative $y$ direction, and we’re also familiar with writing the same vector as a magnitude and direction ($\sqrt{61}$ and $-39.8^\circ$ in this case).

We can do the same thing to represent the relative contributions of resistance and reactance, known as the phase angle. Any complex number of the form $x + iy$ can be written in the form $Ae^{i\theta}$ where $A$ is the amplitude of the number and $\theta$ is the phase. According to Euler, we could also write the phase part of this as

$$e^{i\theta} = \cos \theta + i\sin \theta$$

Notice that there is no frequency information in this form. Also, note that we can extract either the real or imaginary part of this expression to get $\cos \theta$ or $\sin \theta$. Finally, keep in mind that some textbooks (typically those for engineers) use $j$ instead of $i$ to represent $\sqrt{-1}$. One other way to write the complex number $Ae^{i\theta}$ is $A \angle \theta^\circ$. Because the book uses cosines for these functions, if you see something like $75 \angle -12^\circ$, that is equivalent to $75 \cos(\omega t - 12^\circ)$.

It takes a little practice (or a good calculator) to work with complex numbers, but they are very important tools when dealing with reactive circuits. Since the voltage between the two terminals of an AC generator alternates from its positive maximum to its negative minimum and back to the maximum once each cycle, we can imagine the measured voltage as being like the shadow of the hand of a large clock that is illuminated from the 12 o’clock position, as shown below:

As you can see, the length of the shadow will change periodically with the frequency of the sweep. The shadow’s length corresponds to the voltage between the two terminals, and the length of the clock hand is the amplitude of the oscillation. One difference between this model and the phasor concept is that phasors (by convention, of course) rotate counter-clockwise. If we are going to use the cosine representation, we will need the projection of the phasor on the real axis rather than the imaginary one (which would give us the sine term). For an ordinary wall outlet in the US, the magnitude of the
phasor is about 170 V and it rotates at 60 revolutions per second (meaning \( \omega \) is about 377 rad/s).

Being comfortable with both ways of writing complex numbers is important, because if we’re adding/subtracting two or more complex currents or voltages, it will be much easier to put them in \( x + iy \) form and combine the real and imaginary parts separately. On the other hand, if we’re trying to multiply or divide two complex numbers, it is much easier to write them in the form \( Ae^{i\theta} \) and multiply/divide the \( A \)'s while adding or subtracting the phases.

Because we know the functional form of an AC driving voltage or current (i.e., some constant multiplied by \( \cos(\omega t+\phi) \)), we can see that anything requiring us to take the first time derivative of that function will give \(-\omega \sin(\omega t+\phi)\). We could write this with the cosine function instead, using the fact that \( \cos(\theta-90^\circ) = \cos(\theta-\pi/2) = \sin(\theta) \).

Specifically, if we have \( I(t) = I_m \cos(\omega t + \theta) \), and we know that \( V = L \frac{dI}{dt} \) for an inductor, we could find

\[
V(t) = LI_m (-\omega) \cos(\omega t + \theta - 90^\circ)
\]

Of course, since our original current \( I(t) \) could also have been written as \( I_m e^{j\theta} \) (as always, omitting the \( e^{j\omega t} \)) we could also write this as

\[
V(t) = I_m (-\omega) L e^{j\theta} e^{-j \pi / 2}
\]

However, you can use Euler’s formula to quickly see that \( e^{-j \pi / 2} = -j \), leaving us with

\[
V(t) = j \omega L I_m e^{j\theta} = j \omega L I(t)
\]

This leads us to the frequency domain. Now, when we see \( d/dt \), we can replace it with \( j\omega \). Of course, this would have been clear from the first place if we had just written \( I(t) = I_m e^{j\theta} e^{j\omega t} \). This is another benefit of the Euler notation. In angle notation, we can write \( V = \omega L I_m \angle(\theta + 90^\circ) \).

For the case of the capacitor, where the relation is instead \( I = C \frac{dV}{dt} \), the same procedure gives us \( I(t) = j \omega C V(t) \), which we can solve for \( V \) to get

\[
V(t) = \frac{1}{j \omega C} I(t) = \frac{I_m}{\omega C} \angle(\theta - 90^\circ)
\]
This notation makes it clear that the voltage leads the current by 90° in an inductor and it lags the current by 90° in a capacitor. Since the current and voltage are in phase in a resistor, we just get the familiar \( V(t) = I(t) R \) for it.

As you first saw in introductory physics, a useful concept in a non-DC circuit is the **impedance** of the circuit. This is a complex combination of the resistance (the real part, from the resistors) and the reactance (the imaginary part, from any capacitors/inductors). We can then say that the reactance of an inductor is \( X_L = \omega L \) and the reactance of a capacitor is \( X_C = -1/(\omega C) \). The complex impedance is just \( R \) for a resistor, and \( jX_L \) or \( jX_C \) for the inductor or capacitor.

**Series and Parallel RLC Circuits**

If we have a circuit consisting of multiple elements in parallel with one another, the big idea is this: **the voltage is the same across all elements**. On the other hand, if we have a series circuit, the big idea is that **the current is the same throughout the circuit**. Remembering these two things will carry us far.

In introductory physics, you typically only see **series** RLC circuits. In a series circuit, the individual impedances add to produce the final (total) impedance. Since \( X_L \) and \( X_C \) work in opposite directions, their difference gives the total reactive impedance and that is combined with the resistive impedance to get the magnitude of the total in the same way as you have seen earlier

\[
Z = \sqrt{R^2 + (X_L - X_C)^2}
\]

If we kept this in complex form, we would write \( Z = R + i(X_L - X_C) \). Since our resistances are real, our inductive reactances are imaginary and positive, and our capacitive reactances are real and negative, we can see that we get the expression above for \( Z \) by just adding all impedances together.

On the other hand, parallel impedances are combined using the same formula used for resistors in parallel:

\[
\frac{1}{Z_{||}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots
\]

For example, assume we are given a parallel circuit consisting of a 35 \( \Omega \) resistor, a 50 mH inductor, and a 670 \( \mu \text{F} \) capacitor being driven by 180 V oscillating at 60 Hz. First, find \( X_L \) and \( X_C \) using the formulas above. We get \( X_L = 2 \pi f L = 18.85 \Omega \angle 90° \) and \( X_C = 1 / (2 \pi f \ C) = 3.96 \Omega \angle -90° \). The total impedance is then
\[ \frac{1}{Z} = \frac{1}{35\Omega + 0i} + \frac{1}{0 + i18.85\Omega} + \frac{1}{0 - i3.96\Omega} \]

giving us \( Z = 0.7036 - i4.9124 \, \Omega \). Since the voltage across each element is the same, we get the following currents:

\[ I_R = \frac{180V \angle 0^\circ}{35 \, \Omega \angle 0^\circ} = 5.143 \, A \angle 0^\circ \]

Notice that the voltage and current are in phase in the resistor. In the inductor, we get

\[ I_L = \frac{180V \angle 0^\circ}{18.85 \, \Omega \angle 90^\circ} = 9.549 \, A \angle -90^\circ \]

The current is 90° behind the voltage in the inductor. In the capacitor, we get

\[ I_C = \frac{180V \angle 0^\circ}{3.96 \, \Omega \angle -90^\circ} = 45.45 \, A \angle 90^\circ \]

As expected, the current is 90° ahead of the voltage in the capacitor. This gives a total current of 5.143 A + i35.901 A. Notice that we get the same current (verifying Kirchhoff’s current law) if we take the voltage and divide it by the total impedance

\[ I_T = \frac{180V \angle 0^\circ}{4.963 \, \Omega \angle -81.85^\circ} = 36.268 \, A \angle 81.85^\circ \]

If we connect the same elements in a series circuit, we combine the impedances (which are individually unchanged from the parallel case) to get 35 + i14.89 Ω. The current is then

\[ I = \frac{180V \angle 0^\circ}{38.04 \, \Omega \angle 23.05^\circ} = 4.732 \, A \angle -23.05^\circ \]

The voltage across each element is found by Ohm’s law, so we get

\[ V_R = IR = (4.732 \, A \angle -23.05^\circ)(35 \, \Omega \angle 0^\circ) = 165.62V \angle -23.05^\circ \]
If we add these three voltages together as complex quantities, we get 180 V \angle 0^\circ, as we would expect from Kirchoff’s voltage law (which still works). Notice that the resistor’s voltage is in phase with the current, that the capacitor’s voltage is 90° behind it, and that the inductor’s voltage is 90° ahead of the current.

In comparing these two cases, we notice that the current is lower in the series circuit than in the parallel circuit, as we would expect from our work with resistances. It’s also worth pointing out that we’ve gotten rid of a quirk from introductory physics. As you should remember, capacitors in series were combined like resistors in parallel, and vice versa, to find the equivalent capacitance. Inductors and resistors obey the same rules for their combinations, and the capacitors were the “oddball” elements. All impedances we’ve seen work the same as resistors or inductors, however. What has changed? Notice that the impedance of a resistor is just R and the impedance of an inductor is proportional to L. For the capacitor, the impedance is proportional to the inverse of C: 

$$X_C = \frac{1}{(2 \pi f C)}.$$ 

This change has unified the behavior of all three kinds of circuit elements, which makes our work a little simpler.

### Impedance and Delta-to-Wye Transformations

Using the same ideas as in the case of pure resistance, we can find a \( \Delta \) to \( Y \) transformation for impedance.

The transformation to go from a \( \Delta \) to a \( Y \) is
\[ Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \]
\[ Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c} \]
\[ Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \]

while the Y to \(\Delta\) transformation is

\[ Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1} \]
\[ Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2} \]
\[ Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3} \]

**Source Transformations**

Source transformations also work the same way with impedances that they did with pure resistances. A voltage source \(V \angle 0^\circ\) in series with an impedance \(Z\) can be replaced by a current source of magnitude \((V \angle 0^\circ)/Z\) in parallel with the impedance \(Z\). Similarly, a current source \(I \angle 0^\circ\) in parallel with an impedance \(Z\) can be transformed to a voltage source \((I \angle 0^\circ)Z\) in parallel with the impedance \(Z\). We'll do problem 9.41 from your book as an example. It is reproduced below.
The voltage of the sinusoidal source is given as $247.49 \cos(1000t + 45^\circ)$ V and we are asked to find the Thevenin voltage and impedance at terminals a & b. First, we write the voltage in $x + jy$ form. The $45^\circ$ angle means the real and imaginary components are equal in magnitude and are both positive (since $\sin(45^\circ) = \cos(45^\circ)$), so we can write $V = 175 + 175j$.

Next, we find the open-circuit voltage between a & b. To do that, we need to find the current in the circuit, which requires us to find the impedance. The 100 $\Omega$ resistor is in series with the 100 mH inductor, so their impedances are additive. The frequency of the driving voltage is $\omega = 1000$ so the inductor’s impedance is $j1000*100$ mH $= j100 \Omega$, giving this branch a total impedance of 100 + 100 $j$. The capacitor has an impedance of $1/(j\omega C)$, giving us $-j100 \Omega$ in that branch. Combining the two parallel branches into an equivalent impedance gives us $1/[(1/(100+100j)) + 1/(-100j)] = 100 - j100$. When we add this element to the inductor nearest the voltage source (which has the same $j100 \Omega$ impedance as the other inductor) we get a final $Z_{\text{eff}}$ of 100 $\Omega$. Our current leaving the voltage source is then $175 + 175j$ divided by $Z_{\text{eff}}$, giving us $1.75 + 1.75j$ A. The voltage between points a and b is the same as the voltage across the capacitor or the voltage across the combined inductor-resistor leg. We could work from the + terminal downstream to find that voltage, or we could work from the – terminal up, which is what we’ll do. The voltage we get is just equal to the current multiplied by the impedance equivalent to the two branches, which we found earlier to be $100 - j100 \Omega$. That leaves us with a Thevenin voltage of 350 + 0 $j$, or 350 $\angle 0^\circ$.

Next, we need the short-circuit current. This is found more easily by recognizing that a short circuit between a and b eliminates everything in the circuit except the first inductor and the voltage source. The current would therefore be $V/Z$ where $Z$ is the first inductor’s impedance ($j100 \Omega$), so we get a short-circuit current of $1.75 - j1.75$. Finally, we find the Thevenin impedance by using $Z = V_{\text{Th}}/I_{\text{sc}} = (350)/(1.75 - j1.75) = 100 + j100$. We could then redraw the circuit with a voltage source of 350 $\angle 0^\circ$ V connected in series with a 100 $\Omega$ resistor and a 10 mH (meaning $X_L = 100/j\omega$) inductor.

**The Node-Voltage Method**

Unsurprisingly, the node-voltage method that worked so well in purely resistive DC circuits will also work now. As an example, we’ll work problem 9-52, reproduced below.
As explained in the problem we want to use the node-voltage method to find the voltage across the resistor $V_0$. Our reference node will be the one at the bottom of the circuit (2 voltage sources + resistor). The voltages of the sources are given as:

$$V_{g1} = 20 \cos\left(2000t - 36.87^\circ\right) V$$

$$V_{g2} = 50 \sin\left(2000t - 16.26^\circ\right) V$$

We need the voltage magnitudes and phases, but we need them using the cosine convention we have adopted. For $V_{g1}$, we just get $20 \angle -36.87^\circ$. For $V_{g2}$, we need to use the fact that $\sin(x) = \cos(x - 90^\circ)$, allowing us to write $V_{g2}$ as $50 \angle -106.26^\circ$. Converting these to $x + jy$ notation, we would get:

$$V_{g1} = 16 - j12 V$$

$$V_{g2} = -14 - j48 V$$

Now we’re prepared to apply the node-voltage method. The currents leaving the node in question are $(V_0 - V_{g1})/X_L + V_0/10 \Omega + (V_0 - V_{g2})/X_C$. The impedance $X_L = j\omega L = j2 \Omega$ and $X_C = 1 / (j\omega C) = -5j \Omega$. Plugging in the numbers gives us:

$$\left(3.6 - 10.8j\right) = \left(0.1 - 0.3j\right) V_0$$

so $V_0 = 36 \angle 0^\circ$ which we can rewrite as $V = 36 \cos(2000t)$ V.

**The Mesh Current Method**

The mesh current method also works to find currents and voltages in AC circuits just as it does for DC circuits. As an example, we’ll work problem 9-61 from your book (reproduced below).
If the voltage source \( V_g \) is 400 \( \cos(5000 \, \text{t}) \) volts, what is the voltage \( V_0 \) across the 100 \( \Omega \) resistor? Using the mesh current method, we define two counterclockwise loops: the left loop’s current is \( I_a \) and the right loop’s current is \( I_b \). We need to find the impedances \( X_L \) and \( X_C \). We get \( X_L = j \omega \, L = 300 \, j \, \Omega \) and \( X_C = 1/(j \omega \, C) = -100 \, j \, \Omega \). We can then write the following equations

\[
V_g - I_a \, X_L - 50(I_a - I_b) + 150(I_a - I_b) = 0
\]
\[
-150(I_a - I_b) - 50(I_b - I_a) + I_b \, X_C - 100I_b = 0
\]

Solving these two equations for \( I_a \) and \( I_b \) (keeping in mind that \( V_g = 400 \, \angle 0^\circ \)), we get \( I_a = -0.8 - j \, 1.6 \) and \( I_b = -1.6 + j \, 0.8 \). From there, finding the voltage \( V_0 \) in the resistor just requires us to find \( I_b \, R \) which is \(-160 + j \, 80 \). We can also write this as \( 178.89 \, \angle 153.43^\circ \) V.

**Transformers**

In the image below (adapted from an example in the *All About Circuits* PDF), we have a circuit consisting of a driving AC voltage \( V_1 \) which is powering a large inductor \( L_1 \). Inductor \( L_1 \) is coupled to an inductor \( L_2 \) in another circuit. The coupling provides the only source of power in the second circuit (notice that the battery \( V_2 \) is set at zero volts; it’s just there to make PSPICE happy). The first circuit also contains a very small resistance (0.001 \( \Omega \)) and a huge resistance (1 T\( \Omega \)) connecting the two circuits, again for PSPICE. The only important elements in the second circuit are the inductor \( L_2 \) and a load resistance \( R_L \). Because of the magnetic coupling that exists between the two
inductors (specified in PSPICE with a “K” circuit element from the Analog library), a changing current in \(L_1\) will produce a changing current in \(L_2\).

In this circuit, the voltages across the inductors are in phase with one another and with the driving voltage. The current in \(L_2\) is also in phase with this voltage, but the current in \(L_1\) is 180° out of phase with the voltage. The power consumed by \(R_L\) is exactly the same as that consumed by \(L_1\) (since we’ve made a perfect transformer) and the two oscillate together from 0 to their maximum and back at twice the frequency of the AC driving voltage. The power consumed by \(L_2\) also oscillates at the same frequency (twice the voltage source), but 180° out of phase with the power in \(L_1\) and \(R_L\). Also, the power in (or consumed by) \(L_2\) is always negative, moving from 0 to a negative value which is equal to the negative of the power used in \(R_L\). This is reasonable, since the only real elements in the second circuit are \(R_L\) and \(L_2\), so the power used by \(R_L\) must be supplied by \(L_2\).

A mesh-current approach to solving this circuit would include the internal impedance of the source \((Z_1)\), the resistance \(R_1\) associated with the inductance \(L_1\) of the primary coil, the impedance of the load \(Z_L\) (\(R_L\) in our example) and the inductance \(L_2\) and its associated resistance \(R_2\) from the secondary coil. Our equations would be

\[
V_1 - (Z_1 + R_1 + j \omega L_1)I_1 + j \omega M I_2 = 0
\]

\[
j \omega M I_1 - (R_2 + j \omega L_2 + Z_L)I_2 = 0
\]

where \(M\) represents the mutual inductance between the two coils. As we would expect, it is a symmetric term linking the EMF produced in the primary circuit to the current in the secondary circuit and vice-versa. As we have done before, we choose \(I_1\) and \(I_2\) to both be clockwise currents. Following your book, we introduce the simplifying notation
\[ Z_{11} = Z_1 + R_1 + j \omega L_1 \]
\[ Z_{22} = R_2 + j \omega L_2 + Z_L \]

allowing us to write the currents as

\[ I_1 = \frac{Z_{22}}{Z_{11} Z_{22} + \omega^2 M^2} V_1 \]
\[ I_2 = \frac{j \omega M}{Z_{11} Z_{22} + \omega^2 M^2} V_1 = \frac{j \omega M}{Z_{22}} I_1 \]

The effective impedance seen by \( V_1 \) can be found by dividing \( V_1 \) by \( I_1 \) which gives us
\[ Z_{\text{int}} = Z_{11} + (\omega^2 M^2/Z_{22}). \] Subtracting out the internal impedance of the source, \( Z_1 \), we get the impedance at the terminals of the coil \( L_1 \) to be

\[ Z_{\text{term}} = R_1 + j \omega L_1 + \frac{\omega^2 M^2}{R_2 + j \omega L_2 + Z_L} \]

The first two terms are easily seen to be just the impedance of the primary coil itself which can be thought of as an ideal resistance in series with an ideal inductance. The third term is the effect of the secondary circuit on the primary circuit, and is known as reflected impedance.

**Transformer Uses**

The transformer’s purpose is to provide a way to increase voltages, currents or impedances from one coil to the other. The transformer is very much like a lever in this sense: a lever serves as a force multiplier so that you can exert a small force (say, 50 N) at one end and get a large force (500 N, for example) at the other end. Because the lever has no way to provide energy, we know that the price we pay for the force multiplication is a “distance division” since, in our example, moving the 50 N end of the lever by 1 m will cause the 500 N end to move only 10 cm. The total work done by the high-force end is exactly the same (neglecting friction, etc.) as the total work done on the low-force end.

Transformers are similar in that they can be used as voltage multipliers, enabling a 12 V signal to be stepped up to 240 V, for example. The trade-off is that the current on the 240 V side will be only 1/20 of the current on the 12 V side. We can reverse the connections and make a step-down transformer which would take the 12 V signal and
make it $12/20 = 0.6$ V, although that end of the transformer could now supply 20 times the current of the 12 V end.

The transformer can also be considered to be an impedance multiplier/divider. A nice example of this is found in section 9.7 of the “All About Circuits” reference, where two different 1000 W heating elements are examined. One is designed for use with a 250 V circuit while the other is made for a 125 V power supply. By Ohm’s Law, we can see that the high-voltage heating element must have a resistance of $250^2/1000$ or 62.5 Ω while the low-voltage element’s resistance would be $125^2/1000$ or 15.625 Ω. Trying to use these elements in the wrong circuits would not work well. We would generate only $125^2/62.5$ Ω = 250 W of heat in one case, and $250^2/15.625$ Ω = 4000 W (quite probably destroying it) in the other case.

We can imagine using a transformer to convert the 125 V signal to 250 V, or vice versa, as needed. From Faraday’s law of induction, we remember that the EMF induced when the magnetic flux through an $N$-loop coil changes at a rate $d\Phi/dt$ is

$$\xi = -N \frac{d\Phi_B}{dt}$$

If we have two coils and they are linked perfectly (which is not really an attainable goal, but we can get very close to perfect linkage by wrapping each coil around a single iron core since iron is extremely good at directing and confining magnetic flux), they will share a common value of $d\Phi/dt$. That means, if we refer to one coil as the primary and the other as the secondary, we get

$$\frac{\xi_S}{N_S} = -\frac{d\Phi_B}{dt} = \frac{\xi_P}{N_P}$$

or, equivalently, $\xi_S N_P = \xi_P N_S$. This tells us that the ratio of the voltages is equal to the ratio of the number of turns in the coils. This makes sense if we think of each loop being a sort of “pickup” coil: the more loops you have on the iron core, the higher the voltage.

Notice that we could just as easily have said that the two-to-one transformer needed to let us swap heating elements in the previous example was really an impedance transformer. From the point of view of the 125 V supply, it was looking out at a 15.625 Ω load when it was in reality connected to a transformer wired to a 62.5 Ω load. The impedance was transformed by a factor of four, while the voltage/current were only transformed by a factor of two. This is again reasonable since the voltage and current change in opposite directions and their ratio is the impedance. Twice the voltage at half the current gives $V/I = Z = 4$ times the impedance.
This is an important thing to consider in designing audio systems. It’s very important that the speaker’s impedance is matched to the amplifier’s impedance. A mismatch can be fixed by selection of the correct transformer. We already know that the maximum power is transferred from a source to a load when the impedances are equal; the transformer lets that happen.

**Steady-State AC Circuits**

Once transient effects have dissipated in a circuit (something that frequently happens much faster than human reflexes can detect), an RLC circuit will exhibit the properties of a damped (R), driven (AC voltage source) harmonic oscillator (LC). One of the things we’ll want to be able to find is the instantaneous power developed or consumed by a circuit element. This formula is the same as it’s always been: \( P = I \cdot V \). The difference now is that \( I \) and \( V \) are sinusoidally-oscillating functions of time. The current in the circuit will oscillate with the same frequency as the driving voltage, but not necessarily in phase with it. In general, we can write

\[
V(t) = V_m \cos(\omega t + \theta_v)
\]

\[
I(t) = I_m \cos(\omega t + \theta_I)
\]

By convention, we will define the origin of time \( t = 0 \) to be when the current is a maximum, meaning our formulas become

\[
V(t) = V_m \cos(\omega t + \theta_v - \theta_I)
\]

\[
I(t) = I_m \cos(\omega t)
\]

Through the use of various trig identities, we can now write

\[
p = \frac{V_m I_m}{2} (\cos(\theta_v - \theta_I) + \cos(\theta_v - \theta_I) \cos(2\omega t) - \sin(\theta_v - \theta_I) \sin(2\omega t))
\]

Examining these three terms shows that the first does not oscillate in time, but rather provides a constant baseline from which oscillations in power deviate. In the case of simple resistive AC circuits, as you saw in introductory physics, the phases \( \theta_v \) and \( \theta_I \) are equal and this term would be \( \cos 0^\circ = 1 \). In that same case, the second term would be \( \cos(2 \omega t) \) and the third would disappear. This gives a power that oscillates from a minimum of 0 (when \( \cos(2 \omega t) = -1 \)) and a maximum of \( V_m I_m \) (when \( \cos(2 \omega t) = +1 \)). Notice that this oscillation in power has a frequency of 2 \( \omega \), meaning the power oscillates twice for every oscillation of voltage or current.

Notice that a purely inductive or purely capacitive circuit will have phases \( \theta_v \) and \( \theta_I \) which differ by 90°, meaning the first two terms are zero and the final one is \( \pm \sin(2 \omega t) \) (positive for an inductive circuit, negative for a capacitive one). Since this function will
be negative half of the time and positive the other half, the power consumed over one full cycle (or any number of them) is zero. Negative power corresponds to the circuit returning power to the driving voltage; this can only happen if there are reactive elements (inductors or capacitors) in a circuit, since the power consumed by a resistor must be non-negative.

It is sometimes useful to break the above expression for power into a slightly different form:

\[ p = P + P \cos(2\omega t) - Q \sin(2\omega t) \]

where P is known as **average power** or **real power** and Q is the **reactive power**. We can see why if we go back to our earlier case of a purely resistive circuit: Q will be zero and we get that the power \( p = P \left(1 + \cos(2 \omega t)\right) \) (which could also be written as \( p = 2P \cos^2(\omega t) \)). By inspection of the previous formulas, we can see that

\[ P = \frac{V_m}{2} I_m \cos(\theta_v - \theta_l) \]
\[ Q = \frac{V_m}{2} I_m \sin(\theta_v - \theta_l) \]

As mentioned, Q will be positive in an inductive circuit (since \( \theta_v > \theta_l \) and the sine of an angle between 0° and 90° is a positive number) and negative in a capacitive one (since \( \theta_l > \theta_v \) and the sine of an angle between -90° and 0° is a negative number). The units of Q are known as **VARs**, for Volt-Ampere-Reactive to distinguish them from the ordinary Watts used when describing average or real power.

We can consider real power and reactive power to be two legs of a right triangle. The hypotenuse is called **apparent power** and is the magnitude of what is known as the **complex power** \( S \) which has units of Volt-Amperes, or VA.

The three kinds of power are connected to the three things we have measured in \( \Omega \)s (resistance, reactance, and impedance) as shown below

![Diagram](image)

- **Apparent Power**: \( |S| = I^*I \, Z \)
- **Reactive Power**: \( Q = I^*I \, X \)
- **Real Power**: \( P = I^*I \, R \)

\( \phi = \text{Impedance} \)

\( \phi = \text{Phase Angle} \)
There is an important notational change here. Instead of writing $P = I^2R$ as we have seen earlier, we have to take into account the fact that the power may very well be a complex number. In this case, we need to find the absolute magnitude of the complex power. If we write it as $I = x + jy$, we can see that the magnitude of $I$ will be $\sqrt{x^2+y^2}$, or $I^2 = (x^2+y^2)$. We can get the same result if we write $I^*I$ where the asterisk represents the operation of complex conjugation. We could therefore write $I^*I = |I|^2$. Complex conjugation means we just change the sign of the imaginary part of a number. This has no effect on the real part of a number, so we can write a few examples as: $6^* = 6$ \((-3j)^* = (+3j)\) \((-17 + 4j)^* = (-17 - 4j)\).

Similarly, we need the absolute magnitude of $Z$ if this triangle is to work mathematically. Since $Z$ is in general complex and of the form $R \pm jX$, we see that the $Z^*Z$ will again give us a purely real quantity. The magnitude of $Z$ is therefore $\sqrt{Z^*Z} = \sqrt{R^2+X^2}$ (which may be familiar from your introductory physics course).

Returning to the power, $I^*I$ would give $(x-jy)(x+jy) = x^2 - j^2y^2 = x^2 + y^2$. The triangle shows that we need to keep this in mind when working with complex currents. It’s reasonably easy to verify that if $Z = R + jX$, the apparent power will be $I^*I|Z|$, the real power will be $I^*I R$, and the reactive power will be $I^*I X$.

Apparent power is important, because it represents what current-carrying wires have to be designed to carry. As you can see, though, apparent power will be larger than the real power (the power that does work) because of the reactive component. A measure of this mismatch is known as the **power factor**, which is the cosine of the angle $\phi$ shown above. This could also be written as $\cos \phi = P/S$ using basic trigonometry. We could write this in still another form as $\cos(\theta_V - \theta_I)$ which was the first of our three terms describing the power in an AC circuit.

A circuit that is purely resistive (a toaster, for example) will have a power factor of one since it has no reactive or capacitive components (to a good approximation). The impedance $Z$ is just equal to the resistance $R$, meaning $X_L = X_C = 0$ and the reactive power $Q$ is therefore zero. When the $Q$ leg of our right triangle above disappears, we can see that $S = P$ and our power factor $= 1$.

If a circuit is purely reactive, and there is no resistance, we know that $Z = X$ and the real power will be zero. In this case, $S = Q$ and our power factor will be zero.

What if we have a household appliance (perhaps with a large motor) plugged into an outlet providing $120 \text{ V}_{\text{rms}}$ AC at $60 \text{ Hz}$? If it can be represented as an inductance of $0.575 \text{ H}$ in series with a resistance of $125 \\Omega$, what are the powers in this case? First, we can see that $X_L$ will be $2 \pi (60 \text{ Hz}) (0.575 \text{ H})$ or $216.8 \\Omega$. From our earlier work, we could write this as $0 + j216.8 \\Omega$. The resistance is just $125 \\Omega$, and the two elements are in series, giving us $Z = 125 \\Omega + j216.8 \\Omega$.

The voltage source can be represented as $120 \text{ V} \angle 0^\circ$. This gives an rms current $I_{\text{rms}}$ of $0.2395 - j0.4154 \text{ A}$ or $0.4795 \text{ A} \angle -60^\circ$. The voltage drop across the inductor will be
90.06 + j 51.93 V or 103.96 V ∠ 30° (notice that it’s 90° ahead of the current) and 29.94 - j 51.93 V or 59.94 V ∠ -60° (in phase with the current) across the resistor.

The real power P can be expressed as either \( I_{\text{rms}}^* I_{\text{rms}} R \), which gives us 28.74 W, or \( I_{\text{rms}}^* I_{\text{rms}} |Z| \cos \phi = 57.53 \) * \( \cos \phi = 28.74 \) W since \( \phi = \tan^{-1}(X/R) = 60° \). The reactive power would be either \( I_{\text{rms}}^* I_{\text{rms}} X = 49.839 \) VAR or \( I_{\text{rms}}^* I_{\text{rms}} |Z| \sin \phi = 57.53 \) * \( \sin \phi = 49.839 \) VAR. We could write the complex power as \( S = P + jQ = 28.74 + j 49.839 \) VA or \( V_{\text{rms}}^* I_{\text{rms}}^* I_{\text{rms}} Z \) (notice that this is the complex Z, not its magnitude) which gives us the same thing. The magnitude of S, or the apparent power, is then \( \sqrt{P^2 + Q^2} = I_{\text{rms}}^* I_{\text{rms}} |Z| = S^* S = 57.532 \) VA

As an example, the figure below illustrates problem 10-9 in your book. The current source provides a current of 40 Cos 1250 t mA. Find the average, apparent, and reactive powers absorbed by the load.

We can write the current as 0.040 ∠ 0° A (notice that this is \( I_m \), not \( I_{\text{rms}} \)), and we calculate that \( j X_L = j 10,000 \) Ω and \( 1/(j \omega C) = -j 10,000 \) Ω. The resistor and capacitor are in series, so their impedances add to give 5000 Ω -j 10,000 Ω. That impedance is in parallel with the inductor’s impedance of \( j 10,000 \) Ω, so the effective impedance of the whole circuit is 20,000 Ω + j 10,000 Ω. We can then calculate the voltage across the inductor (which is equal to the voltage across the resistor-capacitor branch) by using \( V = I Z \). We get a voltage of 800 + j 400 V.

To find the current in the branch containing the load, we divide the voltage by the impedance of that branch to get \( (800 + j 400 \text{ V})/(5000 \text{ } \Omega - j 10000 \text{ } \Omega) = j 0.080 \text{ A} \) (which we could write as 0.080 ∠90°).

The real power \( P = \frac{1}{2} I^* I R \) (the \( \frac{1}{2} \) is present because we’re using \( I_m \) instead of \( I_{\text{rms}} \)) = 16 W. The reactive power \( Q = \frac{1}{2} I^* I X = -32 \) VAR, and the complex power \( S = P + jQ = \frac{1}{2} I^* I Z = 16 - j 32 \) VA. The apparent power, or magnitude of S is 35.78 W.
Maximum Power Transfer

In the case of a purely resistive circuit, we've already seen that the maximum amount of power is transferred to the load when its resistance matches the internal resistance of the source. When a circuit has a reactive component, things are similar but not identical. First, we reduce the circuit as seen by the load to its Thevenin equivalent voltage in series with its Thevenin impedance. At that point, the total impedance will be \((R_{th} + R_L) + j(X_{th} + X_L)\) so the current (which is the same in both the load and the equivalent impedance) will be

\[
I = \frac{V_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}
\]

The average power consumed by the load is then \(I^* I_R\), or

\[
P = \frac{(V_{th}^* V_{th}) R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}
\]

If we want to find the values for \(R_L\) and \(X_L\) which maximize the average power delivered to the load, we need to take the derivative of \(P\) with respect to these variables and set it equal to zero. As shown in your book, this yields

\[
\frac{\partial P}{\partial X_L} = \frac{-\left(V_{th}^* V_{th}\right) 2 R_L (X_L + X_{th})}{\left[(R_L + R_{th})^2 + (X_L + X_{th})^2\right]^2} = 0
\]

and

\[
\frac{\partial P}{\partial R_L} = \frac{\left(V_{th}^* V_{th}\right) \left[(R_L + R_{th})^2 + (X_L + X_{th})^2\right]^2 - 2 R_L (R_L + R_{th})}{\left((R_L + R_{th})^2 + (X_L + X_{th})^2\right)^2} = 0
\]

For the nontrivial solution (i.e., \(V_{th}\) and \(R_L \neq 0\)), we see from the first equation that the reactance of the load must be the negative of the Thevenin equivalent reactance of the source: \(X_L = -X_{th}\). Under that condition, the second equation requires that \(R_L + R_{th} = 2 R_L\), so \(R_L = R_{th}\). Notice that this reduces to \(Z_L = Z_{th}^*\).

If \(Z\) is not freely adjustable to any value, the best results are to be had when \(X_L\) is as close as possible to \(-X_{th}\) and \(R_L\) is as close as possible to \(\sqrt{R_{th}^2 + (X_L + X_{th})^2}\). If the phase of \(Z\) is restricted instead, maximum power will be transferred to the load when the magnitudes of \(Z_{th}\) and \(Z_L\) are equal.
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